## CPS 140 - Mathematical Foundations of CS Dr. S. Rodger Section: Decidability (handout)

Computability A function f with domain D is computable if there exists some TM M such that M computes f for all values in its domain.

**Decidability** A problem is *decidable* if there exists a TM that can answer yes or no to every statement in the domain of the problem.

## The Halting Problem

Domain: set of all TMs and all strings w.

Question: Given coding of M and w, does M halt on w? (yes or no)

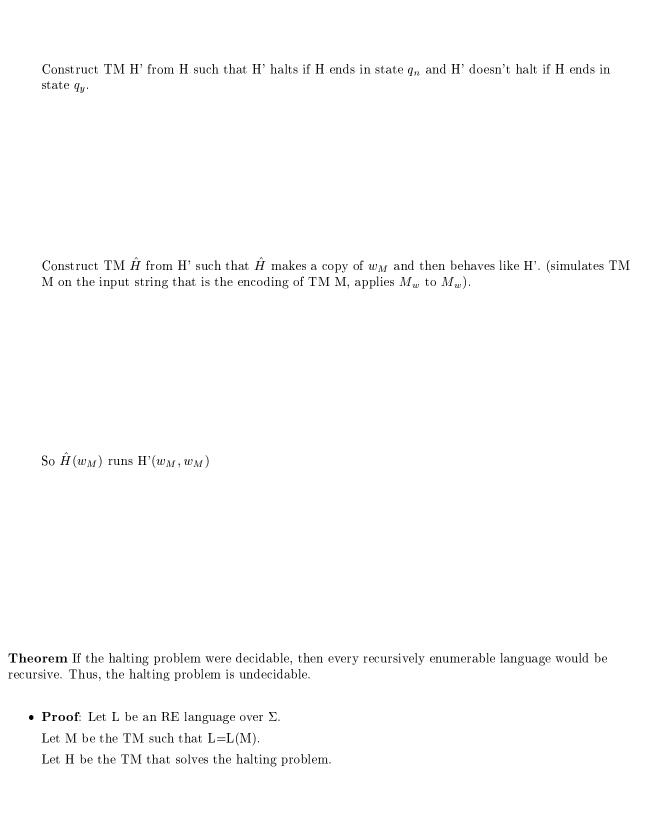
**Theorem** The halting problem is undecidable.

**Proof:** (by contradiction)

• Assume there is a TM H (or algorithm) that solves this problem.

TM H has 2 final states,  $q_y$  represents yes and  $q_n$  represents no.

TM H has input the coding of TM M (denoted  $w_M$ ) and input string w and ends in state  $q_y$  (yes) if M halts on w and ends in state  $q_n$  (no) if M doesn't halt on w.



A problem A is *reduced* to problem B if the decidability of B follows from the decidability of A. Then if we know B is undecidable, then A must be undecidable.

**State-entry problem** Given TM M=(K, $\Sigma$ , , ,  $\delta$ ,  $q_0$ ,  $\square$ ,F), state  $q \in K$ , and string  $w \in \Sigma^*$ , is state q ever entered when M is applied to w?

This is an undecidable problem!

• **Proof:** We will reduce this problem to the halting problem.

Suppose we have a TM E to solve the state-entry problem.

TM E takes as input the coding of a TM M (denoted by  $w_M$ ), a string w and a state q. TM E answers yes if state q is entered and no if state q is not entered.

Construct TM E' which does the following. On input  $w_M$  and w E' first examines the transition functions of M. Whenever  $\delta$  is not defined for some state  $q_i$  and symbol a add the transition  $\delta(q_i, a) = (q, a, R)$ . Let this new state q be the only final state. Let M' be the modified TM. Next, simulate TM E on input  $w_{M'}$ , w and q.

TM E' determines if M halts on w. If M halts on w then TM E' will enter state q in M' and answer yes. If M doesn't halt on w then TM E' will not enter state q, so it will answer no. Since the state-entry problem is decidable, E always gives an answer yes or yes o

But the halting problem is undecidable. Contradiction! Thus, the state-entry problem must be undecidable. QED.

There are some more examples of undecidability in section 5.4.