CPS 140 - Mathematical Foundations of CS
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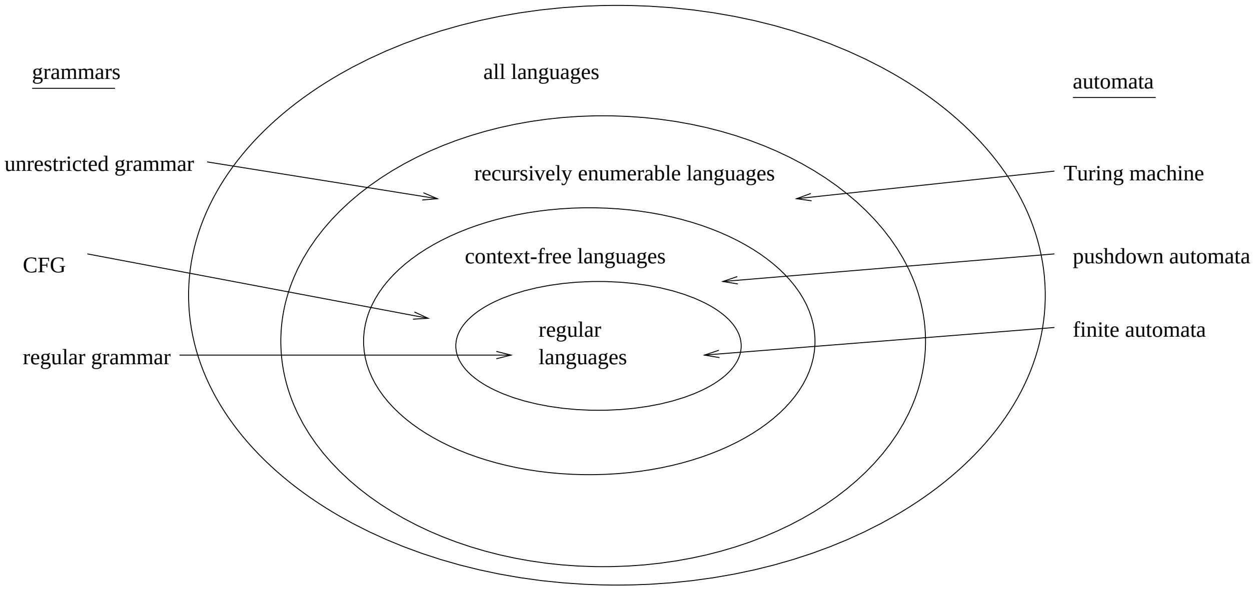
Section: Introduction (Ch. 1) (handout)



Power of Machines

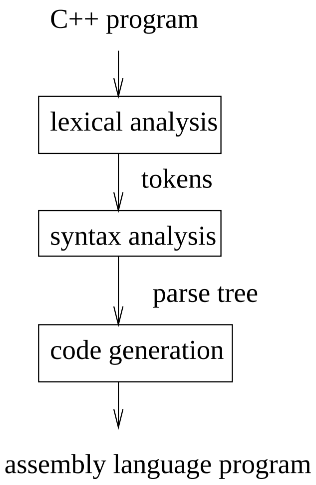
automata	Can do?	Can't do?
FA	integers	arith expr
PDA	arith expr	compute expr
TM	compute expr	decide if halts

Applications

Compiler

- Question: C++ program - is it valid?
- Question: language L, program P - is P valid?

Stages of a Compiler



Set Theory - Read Chapter 1

A Set is a collection of elements.

A={1,4,6,8}, B={2,4,8}, C={3,6,9,12,...}, D={4,8,12,16,...}

- (union) A$\cup$B=
- (intersection) A$\cap$B=
- C$\cap$D=
- (member of) 42 $\in$ C?
- (subset) B$\subset$C?
- B$\cap$A $\subseteq$D?
- (product) A$\times$B=
- $|B|$=
- $\emptyset \in$ B$\cap$C?
- (powerset) 2^B=

Example

Prove: Set S has $2^{|S|}$ subsets.

$\|S\|$	number of subsets
0	
1	
2	
3	
4	

Technique: Proof by Induction

1. Basis: P(1)? Prove smallest instance is true.
2. Induction Hypothesis - I.H.
   Assume P(n) is true for 1,2,...,n
3. Induction Step - I.S.
   Show P(n+1) is true (using I.H.)

Proof of Example:

1. Basis:
2. I.H. Assume
3. I.S. Show

Definition: An infinite set is *countable* if its elements have 1-1 correspondence with the positive integers.

Examples:

- S = { positive odd integers }
- S = { real numbers }
- S = {(i,j) | i,j>0, are integers}

Theorem Let S be an infinite countable set. Its powerset 2^S is not countable.

Proof - Diagonalization

- S is countable, so it's elements can be enumerated.
  S = $\{s_1, s_2, s_3, s_4, s_5, s_6 \ldots\}$
  An element t$\in 2^S$ can be represented by a sequence of 0's and 1's such that the ith position in t is 1 if s_i is in t, 0 if s_i is not in t.

Example, $\{s_2, s_3, s_5\}$ represented by

Example, set containing every other element from S, starting with s_1 is $\{s_1, s_3, s_5, s_7, \ldots\}$ represented by

Suppose 2^S countable. Then we can emunerate all its elements: t_1, t_2, ...

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	...
t_1	0	1	0	1	0	0	1	...
t_2	1	1	0	0	1	1	0	...
t_3	0	0	0	0	1	0	0	...
t_4	1	0	1	0	1	1	0	...
t_5	1	1	1	1	1	1	1	...
t_6	1	0	0	1	0	0	1	...
t_7	0	1	0	1	0	0	0	...
...								

3 Major Concepts

- languages
- grammars
- automata

Languages

- Σ - set of symbols, alphabet
- string - finite sequence of symbols
- language - set of strings defined over Σ

Examples

- $\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
  L=$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, ...\}$
- Σ=$\{a, b, c\}$
  L=$\{ab, ac, cabb\}$
- Σ=$\{a, b\}$
  L=$\{a^n b^n \mid n > 0\}$

Notation

- symbols in alphabet: a, b, c, d, ...
- string names: u,v,w,...

Definition of concatenation

Let w=$a_1 a_2 \dots a_n$ and v=$b_1 b_2 \dots b_m$

Then w∘v OR wv=

See book for formal definitions of other operations.

String Operations

strings: w=abbc, v=ab, u=c

- size of string
  |w|+|v| =
- concatenation
  v^3 = vvv = v∘v∘v =
- v^0 =
- w^R =
- |vv^{R}w|=
- ab ∘ ϵ=

Definition

Σ^* = set of strings obtained by concatenating 0 or more symbols from Σ

Example

$\Sigma = \{a, b\}$

$\Sigma^* =$

$\Sigma^+ =$

Examples

$\Sigma = \{a, b, c\}$, L_1=$\{ab, bc, aba\}$, $L_2 = \{c, bc, bcc\}$

- $L_1 \cup L_2 =$
- $L_1 \cap L_2 =$
- $\overline{L_1} =$
- $\overline{L_1 \cap L_2} =$
- $L_1 \circ L_2 = \{xy \mid x \in L_1 \text{ and } y \in L_2\} =$

Definition

$L^0 = \{\epsilon\}$

$L^2 = L \circ L$

$L^3 = L \circ L \circ L$

$L^* = L^0 \cup L^1 \cup L^2 \cup L^3 \ldots$

$L^+ = L^1 \cup L^2 \cup L^3 \ldots$

Example Is L a countable set?

S = $\{w \in \Sigma^+\}$, $\Sigma = \{a, b\}$

Regular Expressions

Method to represent strings in a language

$+$	union (or)
$\circ$	concatenation (AND) (can omit)
$*$	star-closure (repeat 0 or more times)

Example:

$(a + b)^* \circ a \circ (a + b)^*$

Example:

$(aa)^*$

Definition Given Σ,

1. $\emptyset$ ϵ, $a \in \Sigma$ are R.E.
2. If r and s are R.E. then
   - r+s is R.E.
   - rs is R.E.
   - r* is R.E.
3. r is a R.E. iff it can be derived from (1) with a finite number of applications of (2).

Definition: L(r) = language denoted by R.E. r.

1. $\emptyset$, $\{\epsilon\}$, $\{a\}$ are L denoted by a R.E.
2. if r and s are R.E. then
   (a) L(r+s) = L(r) $\cup$ L(s)
   (b) L(rs) = L(r) $\circ$ L(s)
   (c) L((r)*) = (L(r)*)

Precedence Rules

$*$	highest
$\circ$	
$+$	

Example:

$ab^* + c =$

Examples:

1. $\Sigma = \{a, b\}$, $\{w \in \Sigma^* \mid w$ has an odd number of a's followed by an even number of b's$\}$.

2. $\Sigma = \{a, b\}$, $\{w \in \Sigma^* \mid w$ has no more than 3 a's and must end in $ab\}$.

3. Regular expression for positive and negative integers

Grammars

grammar for english

<sentence> → <subject><verb><d.o.>

<subject> → <noun> | <article><noun>

<verb> → hit | ran | ate

<d.o.> → <article><noun>

<noun> → Fritz | ball

<article> → the | an | a

Examples

Fritz hit the ball.

```
<sentence> -> <subject><verb><d.o>
           -> <noun><verb><d.o>
           -> Fritz <verb><d.o.>
           -> Fritz hit <d.o.>
           -> Fritz hit <article><noun>
           -> Fritz hit the <noun>
           -> Fritz hit the ball
```

The ball hit Fritz.

The ball ate the ball

Syntactically correct?

Semantically correct?

Grammar

G=(V,T,S,P) where

- V - variables (or nonterminals)
- T - terminals
- S - start variable (S$\in$V)
- P - productions (rules)
  x$\rightarrow$y "means" replace x by y
  x$\in$(V$\cup$ T)$^+$, y$\in$(V$\cup$T)*
  where V, T, and P are finite sets.

Definition

w $\Rightarrow$ z	w derives z
w $\overset{*}{\Rightarrow}$ z	derives in 0 or more steps
w $\overset{+}{\Rightarrow}$ z	derives in 1 or more steps

Definition

G=(V,T,P,S)

L(G)={w$\in$T* | S $\overset{*}{\Rightarrow}$ w}

Example

G=({S}, {a,b}, S, P)

P={S$\rightarrow$ aaS, S$\rightarrow$ b}

L(G)=

Example

L(G) = $\{a^n ccb^n \mid n > 0\}$

G =

Automata

Abstract model of a digital computer