Game Theory

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What is Game Theory?

• Settings where multiple agents each have different preferences and set of actions they can take

• Each agent’s utility (potentially) depends on all agents’ actions
  • What is optimal for one agent depends on what other agents do!

• Game theory studies how agents can rationally form beliefs over what other agents will do, and (hence) how agents should act
Penalty Kick Example

Is this a "rational" outcome? If not, what is?
Overview

- Zero-sum games from Adversarial Search lecture
  - Minimax, alpha-beta pruning

- General-sum games

- Normal form vs. Extensive form games
  - Table specifying action-payoff vs. game tree with sequence of actions (and information sets)

- Solving games: dominance, iterated dominance, mixed strategy, Nash Equilibrium
Rock-paper-scissors (zero-sum game)

Row player aka. player 1 chooses a row

Column player aka. player 2 (simultaneously) chooses a column

A row or column is called an action or (pure) strategy

Zero-sum game: the utilities in each entry sum to 0 (or a constant)
Three-player game would be a 3D table with 3 utilities per entry, etc.

<table>
<thead>
<tr>
<th></th>
<th>0, 0</th>
<th>-1, 1</th>
<th>1, -1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, -1</td>
<td>0, 0</td>
<td>-1, 1</td>
<td></td>
</tr>
<tr>
<td>-1, 1</td>
<td>1, -1</td>
<td>0, 0</td>
<td></td>
</tr>
</tbody>
</table>
You could still play a minimax strategy in general-sum games

- pretend that the opponent is only trying to hurt you!

But this is not rational:

<table>
<thead>
<tr>
<th></th>
<th>0, 0</th>
<th>3, 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 0</td>
<td></td>
<td>2, 1</td>
</tr>
</tbody>
</table>

If Column was trying to hurt Row, Column would play Left, so Row should play Down

In reality, Column will play Right (strictly dominant), so Row should play up

not zero-sum
## Chicken

- Two players drive cars towards each other
- If one player goes straight, that player wins
- If both go straight, they both die

![Game Diagram](image)

<table>
<thead>
<tr>
<th></th>
<th>D</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>0, 0</td>
<td>-1, 1</td>
</tr>
<tr>
<td>S</td>
<td>1, -1</td>
<td>-5, -5</td>
</tr>
</tbody>
</table>

Not zero-sum
A “poker-like” game

<table>
<thead>
<tr>
<th></th>
<th>cc</th>
<th>cf</th>
<th>fc</th>
<th>ff</th>
</tr>
</thead>
<tbody>
<tr>
<td>rr</td>
<td>0, 0</td>
<td>0, 0</td>
<td>1, -1</td>
<td>1, -1</td>
</tr>
<tr>
<td>rc</td>
<td>.5, -.5</td>
<td>1.5, -1.5</td>
<td>0, 0</td>
<td>1, -1</td>
</tr>
<tr>
<td>cr</td>
<td>-.5, .5</td>
<td>-.5, .5</td>
<td>1, -1</td>
<td>1, -1</td>
</tr>
<tr>
<td>cc</td>
<td>0, 0</td>
<td>1, -1</td>
<td>0, 0</td>
<td>1, -1</td>
</tr>
</tbody>
</table>
Rock-paper-scissors – Seinfeld variant

MICKEY: All right, rock beats paper! (Mickey smacks Kramer's hand for losing)
KRAMER: I thought paper covered rock.
MICKEY: Nah, rock flies right through paper.
KRAMER: What beats rock?
MICKEY: (looks at hand) Nothing beats rock.
Dominance

• Player i’s strategy $s_i$ strictly dominates $s_i'$ if
  • for any $s_{-i}$, $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$

• $s_i$ weakly dominates $s_i'$ if
  • for any $s_{-i}$, $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$; and
  • for some $s_{-i}$, $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$

-i = “the player(s) other than i”

<table>
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</tr>
</thead>
<tbody>
<tr>
<td>-1, 1</td>
<td>0, 0</td>
<td>-1, 1</td>
<td></td>
</tr>
<tr>
<td>-1, 1</td>
<td>1, -1</td>
<td>0, 0</td>
<td></td>
</tr>
</tbody>
</table>
Back to the poker like game

1 gets King
1 gets Jack

player 1
raise
check
raise
check

player 2
call
fold
call
fold
call

player 1

“nature”

2 1 1 -2 1 -1 1

player 2

call
fold
call
fold
call

cc cf fc ff

rr 0, 0 0, 0 1, -1 1, -1
rc .5, -.5 1.5, -1.5 0, 0 1, -1
cr -.5, .5 -.5, .5 1, -1 1, -1
cc 0, 0 1, -1 0, 0 1, -1
**Prisoner’s Dilemma**

• Pair of criminals has been caught
• District attorney has evidence to convict them of a minor crime (1 year in jail); knows that they committed a major crime together (3 years in jail) but cannot prove it

• Offers them a deal:
  – If both confess to the major crime, they each get a 1 year reduction
  – If only one confesses, that one gets 3 years reduction

<table>
<thead>
<tr>
<th></th>
<th>Confess</th>
<th>Don’t Confess</th>
</tr>
</thead>
<tbody>
<tr>
<td>Confess</td>
<td>-2, -2</td>
<td>0, -3</td>
</tr>
<tr>
<td>Don’t Confess</td>
<td>-3, 0</td>
<td>-1, -1</td>
</tr>
</tbody>
</table>
Iterated Dominance

- Iterated dominance: remove (strictly/weakly) dominated strategy, repeat
- Iterated strict dominance on Seinfeld’s RPS:

\[
\begin{array}{ccc}
0, 0 & 1, -1 & 1, -1 \\
-1, 1 & 0, 0 & -1, 1 \\
-1, 1 & 1, -1 & 0, 0 \\
\end{array}
\]
“2/3 of the average” game

- Everyone writes down a number between 0 and 100
- Person closest to 2/3 of the average wins

Example:
- A says 50
- B says 10
- C says 90
- Average(50, 10, 90) = 50
- 2/3 of average = 33.33
- A is closest (|50-33.33| = 16.67), so A wins

Try?
"2/3 of the average" via dominance

\[ \left( \frac{2}{3} \right) \times 100 \]

\[ \left( \frac{2}{3} \right) \times \left( \frac{2}{3} \right) \times 100 \]

\[ \ldots \]

\[ 0 \]

\{ dominated \}

\{ dominated after removal of (originally) dominated strategies \}
Mixed strategy

- **Mixed strategy** for player $i = \text{probability distribution over player i’s (pure) strategies}
- E.g. $1/3$, $1/3$, $1/3$
- Example of dominance by a mixed strategy:

<table>
<thead>
<tr>
<th></th>
<th>3, 0</th>
<th>0, 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0, 0</td>
<td>3, 0</td>
<td></td>
</tr>
<tr>
<td>1/2</td>
<td>1, 0</td>
<td>1, 0</td>
</tr>
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</table>
Best-Response

- Let $A$ be a matrix of player 1’s payoffs
- Let $s_2$ be a mixed strategy for player 2
- $As_2 = \text{vector of expected payoffs for each strategy for player 1}$
- Highest entry indicates **best response** for player 1
- Any mixture of ties is also BR
- Generalizes to >2 players

\[
\begin{array}{cc}
0, 0 & -1, 1 \\
1, -1 & -5, -5 \\
\end{array}
\]

$\sigma_2$
Nash Equilibrium
[Nash 50]

- A vector of strategies (one for each player) = a strategy profile
- Strategy profile \((\sigma_1, \sigma_2, \ldots, \sigma_n)\) is a Nash equilibrium if each \(\sigma_i\) is a best response to \(\sigma_{-i}\)
- Does not say anything about multiple agents changing their strategies at the same time
- In any (finite) game, at least one Nash equilibrium (possibly using mixed strategies) exists [Nash 50]
(D, S) and (S, D) are Nash equilibria
- They are pure-strategy Nash equilibria: nobody randomizes
- They are also strict Nash equilibria: changing your strategy will make you strictly worse off

No other pure-strategy Nash equilibria
• (D, S) and (S, D) are Nash equilibria
• Which do you play?
• What if player 1 assumes (S, D), player 2 assumes (D, S)
• Play is (S, S) = (-5, -5)!!

• This is the equilibrium selection problem
Any pure-strategy Nash equilibria?
But it has a **mixed-strategy Nash equilibrium**:
Both players put probability 1/3 on each action
If the other player does this, every action will give you expected utility 0
  – Might as well randomize

<table>
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<th>Paper</th>
<th>Scissors</th>
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<td><strong>Rock</strong></td>
<td>0, 0</td>
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<td>0, 0</td>
<td>-1, 1</td>
</tr>
<tr>
<td><strong>Scissors</strong></td>
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Is there a Nash equilibrium that uses mixed strategies -- say, where player 1 uses a mixed strategy?

If a mixed strategy is a best response, then all of the pure strategies that it randomizes over must also be best responses.

So we need to make player 1 indifferent between D and S.

Player 1’s utility for playing D = -p^c_S

Player 1’s utility for playing S = p^c_D - 5p^c_S = 1 - 6p^c_S

So we need -p^c_S = 1 - 6p^c_S which means p^c_S = 1/5

Then, player 2 needs to be indifferent as well.

Mixed-strategy Nash equilibrium: ((4/5 D, 1/5 S), (4/5 D, 1/5 S))

- People may die! Expected utility -1/5 for each player
The “poker-like game” again

- To make player 1 indifferent between rr and rc, we need:
  utility for rr = 0*P(cc)+1*(1-P(cc)) = .5*P(cc)+0*(1-P(cc)) = utility for rc
  That is, P(cc) = 2/3
- To make player 2 indifferent between cc and fc, we need:
  utility for cc = 0*P(rr)+(-.5)*(1-P(rr)) = -1*P(rr)+0*(1-P(rr)) = utility for fc
  That is, P(rr) = 1/3
Computational considerations

- Zero-sum games - solved efficiently as LP
- General sum games may require exponential time (in # of actions) to find a single equilibrium (no known efficient algorithm and good reasons to suspect that none exists)

- Some better news: Despite bad worst-case complexity, many games can be solved quickly
Extensions

• Partial information
• Uncertainty about the game parameters, e.g., payoffs (Bayesian games)
• Repeated games: Simple learning algorithms can converge to equilibria in some repeated games
• Multistep games with distributions over next states (game theory + MDPs = stochastic games)
• Multistep + partial information (Partially observable stochastic games)

• Game theory is so general, that it can encompass essentially all aspects of strategic, multiagent behavior, e.g., negotiating, threats, bluffs, coalitions, bribes, etc.