Monte Carlo alg for area of a circle

Let \( X_i = \begin{cases} 0 & \text{i-th point is not in circle} \\ 1 & \text{i-th point is in circle} \end{cases} \)

\[
\mathbb{E}(X_i) = \Pr(X_i = 0) = \frac{\text{area of circle}}{\text{area of square}} = \frac{\pi}{4}
\]

Final output: \( X = \sum_{i=1}^{n} X_i \) (\( X \) counts in alg)

\[
\frac{4}{n} \sum_{i=1}^{n} \mathbb{E}(X_i) = \frac{4}{n} \sum_{i=1}^{n} \frac{\pi}{4} = \frac{4 \pi}{4} = \pi
\]

Variance

\[
\text{Var}(X_i) = \mathbb{E}[(X_i - \mathbb{E}(X_i))^2] = \mathbb{E}[(X_i - \frac{\pi}{4})^2]
\]

\[
= \mathbb{E}(X_i - \frac{\pi}{4})^2 = \mathbb{E}(X_i - \frac{\pi}{4})^2 + \Pr(\text{point is inside circle})
\]

\[
= \frac{\pi}{4} + \Pr(\text{point is inside circle})
\]

Recall: \( X = \sum_{i=1}^{n} X_i = X_1 + X_2 + \ldots + X_n \)

\[
\text{Var}(X) = \sum_{i=1}^{n} \text{Var}(X_i) = n \text{Var}(X)
\]

\[
\text{Var output} = \frac{4}{n} \cdot \text{Var}(X) = \frac{16 \pi}{4n} = \frac{4 \pi}{n}
\]

Claim: \( \Pr(\text{output - area of circle} > \frac{4}{\sqrt{n}} \leq \frac{1}{4} \)

(in other words, with probability \( \frac{3}{4} \), output is within \( \frac{4}{\sqrt{n}} \))

\[
\sqrt{\frac{\text{Var output}}{n}} \leq \frac{2}{\sqrt{n}}
\]

\[
2 \sqrt{\text{Var output}} \leq \frac{4}{\sqrt{n}}
\]

\[
\text{Pr}(\text{output} > 2 \sqrt{\text{Var output}}) \leq \frac{1}{4}
\]
\[
2 \sqrt{\text{Var(output)}} \leq \frac{4}{\sqrt{n}}
\]

By Chebyshev's inequality, \( \Pr\left[|\text{output} - \mathbb{E}[	ext{output}]| > 2 \sqrt{\text{Var(output)}}\right] \leq \frac{1}{4} \) (choosing \( \lambda = 2 \) in the inequality).

\[
\Pr\left[|\text{output} - \text{area}| > \frac{4}{\sqrt{n}}\right] \leq \Pr\left[|\text{output} - \mathbb{E}[\text{output}]| > 2 \sqrt{\text{Var(output)}}\right] \leq \frac{1}{4}
\]

- **Hashing**
  - **fixed hash function**
    - \( a[m] \)
    - \( f(i) = i \mod m \)
    - if the set is \( S = \{5, m+5, 2m+5, \ldots, 2n+m+5\} \)
    - for all \( i \in S \), \( f(i) = 5 \)

\[
\begin{array}{ccccccc}
0 & 1 & 2 & 3 & 4 & 5 \\
\downarrow & & & & & \\
5 & 6 & 7 & 8 & 9 & 10
\end{array}
\]

- **random function**
  - how to choose a random hash function
  - for every integer \( i \), choose \( f(i) \) independently at random in \( \{0, 1, 2, \ldots, m-1\} \)
  - (function is chosen at the beginning, and fixed for later use)

- **hash function by modular arithmetic**
  - \( f_{a,b}(x) = ax + b \pmod{p} \)
  - \( \Pr_{ab}[f_{a,b}(x) = u, f_{a,b}(y) = v] = \frac{1}{p^2} \)

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Proof:  
In order to have $f_{ab}(x) = u \Rightarrow a, b(y) = v$

\[
\begin{cases}
    a(x) + b = u \pmod{p} \\
    a(y) + b = v \pmod{p}
\end{cases}
\]

System of linear equation has a unique solution

\[
a = \frac{(u-v)}{(x-y)} \quad b = u - ax = v - ay
\]

$Pr_{a,b}[f(x) = u, f(y) = v] = \frac{1}{p^2}$  

$p^2$ choices for $a, b$

Pseudo-code for hashing

initialize:
choose a prime number $p$
choose $a, b \in \{0, 1, \ldots, p-1\}$ randomly.
Create an array $a[0 \ldots p-1]$  
\[\text{the size of array can also be } m < p\]
initialize each cell $a[i]$ to be the head of an empty linked list.

\[
f(x)
\]
\[
\begin{cases}
    \text{return } (ax+b) \pmod{p} & \text{if the size of array is } m, \\
    \text{return } ((ax+b) \pmod{p}) \pmod{n}
\end{cases}
\]

insert ($x$)
insert $x$ to the linked list $a[f(x)]$

find ($x$)
check if $x$ is in the linked list $a[f(x)]$