CompSci 102 Discrete Mathematics for CS
Spring 2005 Forbes Sample Test 1 Questions

1. (20 pts.) Satisfiabilty For each of the following Boolean expressions, decide whether it is (i) valid, (ii) satisfiable, or (iii) unsatisfiable. Give *all* applicable properties with justifications.

(a) $A \wedge \neg A \wedge \neg B$

(b) $(A \Rightarrow B) \wedge (B \rightarrow C) \wedge (C \rightarrow \neg A)$

(c) $(A \Rightarrow B) \vee (B \Rightarrow A)$

2. (Each 10 pts.) Logic and Proofs

(a) Can you define *open sentences* (i.e., sentences whose truth value depends on some variable x) $P(x)$ and $Q(x)$ and a universe U so that

$(\forall x \in U)(P(x) \Rightarrow Q(x))$ is false, and

$(\forall x \in U)(Q(x) \Rightarrow P(x))$ is false?

If yes, give an example. If no, explain why not.

(b) Prove that for all x in $\mathcal{R}$, if $\sqrt{2}+x$ is rational, then x is irrational. What proof technique did you use?

3. (5 points each pts.) Conditionals

a. Can you define *propositions* (i.e. sentences with a fixed truth value) P and Q so that

$P \Rightarrow Q$ is false, and

$Q \Rightarrow P$ is false? If yes, give an example. If no, explain why not.

b. Can you define *open sentences* (that is, sentences whose truth value depends on some variable x) $P(x)$ and $Q(x)$ and a universe U so that

$(\forall x \in U)(P(x) \Rightarrow Q(x))$ is false, and

$(\forall x \in U)(Q(x) \Rightarrow P(x))$ is false? If yes, give an example. If no, explain why

not.

4. (5 pts.) Mod

(a) Is $175 \equiv 22 (\text{mod} 17)$?

(b) Use the Euclidean algorithm to find gcd(34,21)

5. (15 pts.) Numbers

(a) Prove that there are no solutions in the positive integers to the equation $x^4 + y^4 = 100$.

(b) Prove that

$$\sum_{j=n}^{2n-1} (2j+1) = 3n^2$$

whenever n is a positive integer

(c) Suppose that a_n is defined recursively by $a_n = a_{n-1}^2 - 1$ and that $a_0 = 2$. Find a_3 and a_4.

6. (15 pts.) Induction:

For every $n \in \mathbf{N}$ let $P(n)$ be a statement about n. Suppose that $P(13)$ is false, and for every $n \in \mathbf{N}$, $P(n) \Rightarrow P(n+1)$. What can we conclude about $P(1)$? Why?

7. (10 pts.) Proof to Grade

What is wrong with the following induction proof?

Claim: $(\forall n \in \mathbf{N})(n^2 \leq n)$.

Proof:

i) Base Case: When $n = 1$, the statement is $1^2 \leq 1$ which is true.

ii) Inductive step: Now suppose that $k \in \mathbf{N}$, and $k^2 \leq k$.

We need to show that $(k+1)^2 \leq k+1$

Working backwards we see that:

$$(k+1)^2 \leq k+1 \quad (1)$$

$$k^2 + 2k + 1 \leq k+1 \quad (2)$$

$$k^2 + 2k \leq k \quad (3)$$

$$k^2 \leq k \quad (4)$$

So we get back to our original hypothesis which is assumed to be true. Hence, for every $n \in \mathbf{N}$ we know that $n^2 \leq n$. ♡

8. (20 pts.) Fibonacci numbers

Recall that the Fibonacci numbers are defined by $F(0) = 0$, $F(1) = 1$ and for all $n \geq 2$, $F(n) = F(n-1) + F(n-2)$. Prove by induction that $\sum_{i=k}^{n} F(i) = F(n+2) - F(k+1)$.

9. (15 pts.) Tiling

Let D_n be the number of ways to tile a $2 \times n$ checkerboard with dominos, where a domino is a 1×2 piece. Prove that $D_n \leq 2^n$ for all positive integers n. (HINT: find a recurrence relation.)