Test 1 Solutions: CompSci 102

PROBLEM 1 : (***Logic (16 pts.)***)

A. Proof by enumeration

p	r	s	$p \Leftrightarrow r$	$r \Leftrightarrow s$	$((p \Leftrightarrow r) \Leftrightarrow s)$	$(p \Leftrightarrow (r \Leftrightarrow s))$
F	F	F	T	T	F	F
F	F	T	T	F	T	T
F	T	F	F	F	T	T
F	T	T	F	T	F	F
T	F	F	F	T	T	T
T	F	T	F	F	F	F
T	T	F	T	F	F	F
T	T	T	T	T	T	T

The last two columns are equivalent, so the associative law holds for $\Leftrightarrow$.

B. Let $D(x, y)$ be the statement "x can dunk on y," where the universe of discourse is the set of all people in the world. Use quantifiers to express each of the following statements:

1. $\forall x \exists y D(x, y)$
2. $\forall x \exists y D(y, x)$
3. $\neg \exists x (D(x, Amare) \wedge D(x, Yao))$
4. $\exists x \exists y \forall z (D(Spud, x) \wedge D(Spud, y) \wedge (x \neq y) \wedge (((z \neq x) \wedge (z \neq y)) \Leftarrow \neg D(Spud, z)))$
5. $\exists x \forall y \forall z \exists r (D(y, x) \wedge ((x \neq z) \Leftarrow \neg D(x, z)))$

PROBLEM 2 : (***Numbers (16 pts)***)

A. True

$$\sum_{i=1}^{2n} (i^2 + 1) \leq \sum_{i=1}^{2n} (n^2 + n) = 2n(n^2 + n) = 2n^3 + 2n^2 < kn^3$$

for $k = 100$ and $n > 100$. Thus $f(n) = \sum_{i=1}^{2n} (i^2 + 1) \in O(n^3)$

$$\sum_{i=1}^{2n} (i^2 + 1) = \sum_{i=1}^{2n} i^2 + \sum_{i=1}^{2n} 1 = 2n + \sum_{i=1}^{2n} i^2 > \sum_{i=0}^{2n} i^2$$

From the book (sum of quadratics), we know

$$\sum_{i=0}^{2n} i^2 = \frac{2n(2n+1)(4n+1)}{6}$$

Therefore,

$$\frac{2n(2n+1)(4n+1)}{6} > kn^3$$

for $k = 1$ and $n > 2$. Thus $f(n) \in \Omega(n^3)$

Since $f(n) \in O(n^3)$ and $f(n) \in \Omega(n^3)$, then $f(n) \in \Theta(n^3)$

B. Since $a \equiv b (\text{mod } m)$, $m | a - b$. Hence there is an integer k such that $a - b = mk$. It follows that $(a - c) - (b - c) = a - b = mk$, so $a - c \equiv b - c (\text{mod } m)$.

C. Let $b \in B$ and $a = f^{-1}(b)$, where $a \in A$.

$f(a) = b$	definition of inverse
$a = f^{-1}(b)$	premise
$(f \circ f^{-1})(b) = f((f^{-1}(b)))$	definition of composition
$f((f^{-1}(b))) = f(a) = b$	premises
$\forall b \in B, (f \circ f^{-1})(b) = b$	previous steps

Doing the other case, let $a \in A$ and $b = f(a)$ where $b \in B$.

$f^{-1}(b) = a$	definition of inverse
$(f^{-1} \circ f)(a) = f{-}1(f((a))) = f{-}1(b) = a$	defn of composition, premise
$\forall a \in A, (f^{-1} \circ f)(a) = a$	previous steps

Both cases of the consequent are proven.

PROBLEM 3 : (***Proofs (8 pts.)***)

A. In the inductive step, the proof for $P(n+1)$ appeals to $P(n)$ and $P(n-1)$, which fails for $n+1=2$ because $P(0)$ is unproven and false.

B. The step where we break up the strings is incorrect. For example, a string of length 1 is not the pasting together of two strings of length 0.

PROBLEM 4 : (***Sets (16 pts)***)

A. True:

Logical derivation:

$$\begin{aligned}
A - (B \cap C) &= x | (x \in A) \wedge \neg (x \in (B \cap C)) \\
&= x | (x \in A) \wedge \neg ((x \in B) \wedge (x \in C)) \\
&= x | (x \in A) \wedge (\neg (x \in B) \vee \neg (x \in C)) \\
&= x | ((x \in A) \wedge \neg (x \in B)) \vee ((x \in A) \wedge \neg (x \in C)) \\
&= x | x \in (A - B) \vee x \in (A - C) \\
&= (A - B) \cup (A - C)
\end{aligned}$$

or, a clear answer in English is also acceptable:

The elements of $A - (B \cap C)$ are, by definition, in A, but not in both B and C. Therefore they are in A, and either not in B, or not in C. Therefore they are either in A but not B, or A but not C. So they are in the union of $(A - B)$ and $(A - C)$.

The converse of the above argument is also true by the same reasoning.

B. Use strong induction, assume $A_1 \cap (A_2 \cup \ldots \cup A_n) = (A_1 \cap A_2) \cup \ldots \cup (A_1 \cap A_n)$ true for $n \geq 3$ show for $n+1$.

$P(3)$ is true because that is the normal distributive law for intersection over union.

Inductive hypothesis: $P(3) \wedge \ldots \wedge P(n) \Rightarrow P(n+1)$

$$\begin{aligned} A_1 \cap (A_2 \cup \ldots \cup A_{n+1}) &= A_1 \cap ((A_2 \cup \ldots \cup A_n) \cup A_{n+1}) \\ &= [A_1 \cap (A_2 \cup \ldots \cup A_n)] \cup (A_1 \cap A_{n+1}) \\ &= [(A_1 \cap A_2) \cup \ldots \cup (A_1 \cap A_n)] \cup (A_1 \cap A_{n+1}) \\ &= (A_1 \cap A_2) \cup \ldots \cup (A_1 \cap A_{n+1}) \end{aligned}$$

PROBLEM 5 : (***Recursion (19 points)***)

A. Fibonnaci questions

1. $\langle 0, 0, 1, 2, 4, 7 \rangle$.
2. We claim that $T_n = F_{n+1} - 1$. We can prove this by strong induction on n.
   - Base case: $P(0)$ is clearly true, since $T_0 = 0$, and $(F_1 - 1) = 1 - 1 = 0$.
   - Inductive step: prove $P(1) \wedge \ldots \wedge P(n) \Rightarrow P(n+1)$ for all $n \geq 0$.
   (a) The inductive hypothesis is that $T_i = F_{i+1} - 1$ for $0 \leq i \leq n$.
   (b) To prove: $T_{n+1} = F_{n+2} - 1$.
   (c) Clearly, $T_{n+1} = T_n + T_{n-1} + 1$. Thus

$$\begin{aligned} T_{n+1} &= T_n + T_{n-1} + 1 \\ &= (F_{n+1} - 1) + (F_n - 1) + 1 \qquad \text{by the induction hypothesis} \\ &= F_{n+1} + F_n + 1 \\ &= F_{n+2} + 1 \qquad \text{by the definition of the } F \text{ numbers} \end{aligned}$$

B. Consider the problem of forming arbitrary postage from combinations of certain stamps.

1. We can form any amount ≥ 40 cents using 5 and 11 cent stamps.
2. Let $P(n)$ be the proposition that we can form n cents of postage using 5 and 11 cent stamps for $n \geq 40$. Let us define $Q(n)$ to be the proposition that we can form postage worth $n, n+1, \ldots, n+4$ cents, i.e., $Q(n) = P(n) \wedge P(n+1) \wedge \ldots \wedge P(n+4)$. The proof will be by simple induction and will prove $\forall n \geq 40\ Q(n)$. Since $Q(n) \Rightarrow P(n)$, we will have proved $\forall n \geq 40\ P(n)$.
   - Base case: We can $40 + i$ cents (for $0 \leq i \leq 4$) by $8 - 2i$ stamps of 5 cents, and i stamps of 11 cents. Hence $Q(40)$ is true.
   - Inductive step: prove $Q(n) \Rightarrow Q(n+1)$ for all natural numbers $n \geq 40$.
   (a) The inductive hypothesis states that, for all natural numbers m from 40 to $n+4$, m c of postage can be composed from 5c and 11c stamps.
   (b) To prove: $Q(n+1) = P(n+1) \wedge \ldots \wedge P(n+5)$. Since we know that the first 4 propositions are true, we need to just prove $P(n+5)$, i.e., $(n+5)$c of postage can be composed from 5c and 11c stamps, where $n \geq 40$.
   (c) The inductive hypothesis $Q(n) \Rightarrow P(n)$. If nc can be composed from 5c and 11c stamps, then $(n+5)$c can be composed from 5c and 11c stamps simply by adding one more 5c stamp. This proves $P(n+5)$, and hence $Q(n+1)$.
3. The proof will be by strong induction, and will try to prove $\forall n \geq 40\, P(n)$. (Note that the content of this proof is very similar to that of the previous part, but it can be stated much more simply.
   - Base case: We can get 40 cents by 8 stamps of 5 cents.
   - Inductive step: prove $P(40) \wedge \ldots \wedge P(n) \Rightarrow P(n+1)$ for all natural numbers $n \geq 40$.
   (a) The inductive hypothesis states that, for all natural numbers m from 40 to n, m c of postage can be composed from 5c and 11c stamps.

(b) To prove: $(n+1)$c of postage can be composed from 5c and 11c stamps.

(c) The cases where $(n+1)$ is between 41 and 44 cents, we can do things thus: for $0 < i < 5$, $40 + i$ cents can be obtained by $8 - 2i$ stamps of 5 cents, and i stamps of 11 cents.

(d) For all natural numbers $n + 1 > 44$, the inductive hypothesis entails the proposition $P(n-5)$. If $(n-5)$c can be composed from 5c and 11c stamps, then $(n+1)$c can be composed from 5c and 11c stamps simply by adding one more 5c stamp.

Extra Credit (max 5 pts.)

- Let $\gcd(a, b) = g$, and let $\alpha = a/g, \beta = b/g$. Then any amount of the form $g(\alpha - 1)(\beta - 1)$ can be formed using a and b cent stamps. XXX Proof later.

- Assume there were $2n$ people (including host and hostess) in the party. When the host asked the question he must have got $2n - 1$ responses (including from his wife) and all of the responses were different.

  The responses have to be $(0, 1, 2, 3, \ldots, 2n-2)$ to satisfy the above requirements. As $2n - 2$ is the maximum possible handshakes any person in this party could have made. Let $P(x)$ mean a person shook x hands and H be the host.

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H: <-------->2n-2   0
           2n-3        1
          2n-4          2
          2n-5          3
          .             .
           .           .
             .       .
              n  n-1 n-2
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  There are $2n - 1$ on the circle.

  $P(2n-2)$ must have handshaked with H (because in the circle he can handshake with only $2n - 3$. He has to exclude himself also.) $P(2n-3)$ must have handshaked with H (because in the circle he can handshake with only $2n - 4$.) $P(2n-4)$ must have handshaked with H (because in the circle he can handshake with only $2n - 5$.) $P(n)$ must have handshaked with H (because in the circle he can handshake with only $n-1$.) From $P(n-1)$ to $P(0)$ nobody handshakes with H, because, for them the handshake numbers are satisfied on the circle itself. This leaves H with $n - 1$ handshakes.

  $P(0)$ must be the spouse of $P(2n-2)$ (since $P(2n-2)$ handshakes with everbody else.)

$$\vdots$$

  same logic till $P(n-2)$. The Hostess is left to be $P(n-1)$.

  Thus if n is the number of couples including the host couple, the Host made $(n-1)$ handshakes and the Hostess - made $(n-1)$ handshakes.