

Today's topics

- Functions
 - Notations and terms
 - One-to-One vs. Onto
 - Floor, ceiling, and identity
- Reading: Sections 1.8
- Upcoming
 - Algorithms

On to section 1.8... Functions

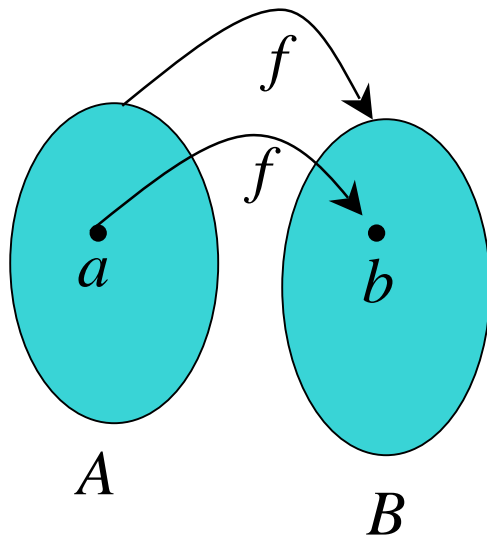
- From calculus, you are familiar with the concept of a real-valued function f , which assigns to each number $x \in \mathbf{R}$ a particular value $y = f(x)$, where $y \in \mathbf{R}$.
- But, the notion of a function can also be naturally generalized to the concept of assigning elements of *any* set to elements of *any* set. (Also known as a *map*.)

Function: Formal Definition

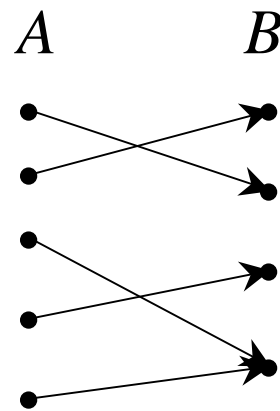
- For any sets A, B , we say that a *function* f from (or “mapping”) A to B ($f:A \rightarrow B$) is a particular assignment of exactly one element $f(x) \in B$ to each element $x \in A$.
- Some further generalizations of this idea:
 - A *partial* (non-total) function f assigns zero or one elements of B to each element $x \in A$.
 - Functions of n arguments; relations (ch. 6).

Graphical Representations

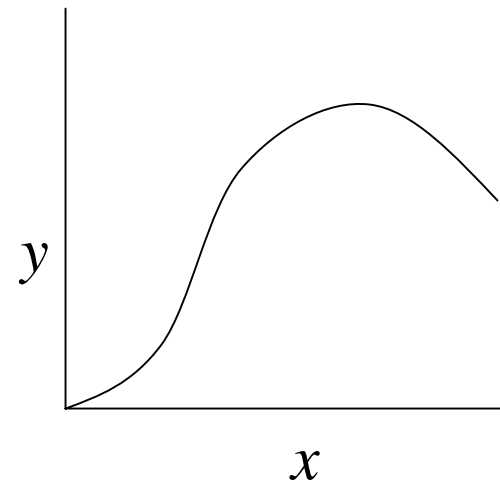
- Functions can be represented graphically in several ways:



Like Venn diagrams



Bipartite Graph



Plot

Functions We've Seen So Far

- A *proposition* can be viewed as a function from “situations” to truth values $\{\mathbf{T}, \mathbf{F}\}$
 - A logic system called *situation theory*.
 - $p = \text{“It is raining.”}$; $s = \text{our situation here, now}$
 - $p(s) \in \{\mathbf{T}, \mathbf{F}\}$.
- A *propositional operator* can be viewed as a function from *ordered pairs* of truth values to truth values: *e.g.*, $\vee((\mathbf{F}, \mathbf{T})) = \mathbf{T}$.

Another example: $\rightarrow((\mathbf{T}, \mathbf{F})) = \mathbf{F}$.

A Neat Trick

- Sometimes we write Y^X to denote the set F of *all* possible functions $f: X \rightarrow Y$.
- This notation is especially appropriate, because for finite X, Y , we have $|F| = |Y|^{|X|}$.
- If we use representations $\mathbf{F} \equiv \mathbf{0}$, $\mathbf{T} \equiv \mathbf{1}$, $\mathbf{2} := \{\mathbf{0}, \mathbf{1}\} = \{\mathbf{F}, \mathbf{T}\}$, then a subset $T \subseteq S$ is just a function from S to $\mathbf{2}$, so the power set of S (set of all such fns.) is $\mathbf{2}^S$ in this notation.

Some Function Terminology

- If it is written that $f:A \rightarrow B$, and $f(a)=b$ (where $a \in A$ & $b \in B$), then we say:
 - A is the *domain* of f .
 - B is the *codomain* of f .
 - b is the *image* of a under f .
 - a is a *pre-image* of b under f .
 - In general, b may have more than 1 pre-image.
 - The *range* $R \subseteq B$ of f is $R = \{ b \mid \exists a f(a)=b \}$.

We also say
the *signature*
of f is $A \rightarrow B$.

Range versus Codomain

- The range of a function might *not* be its whole codomain.
- The codomain is the set that the function is *declared* to map all domain values into.
- The range is the *particular* set of values in the codomain that the function *actually* maps elements of the domain to.

Range vs. Codomain - Example

- Suppose I declare to you that: “ f is a function mapping students in this class to the set of grades $\{A,B,C,D,E\}$.”
- At this point, you know f 's codomain is _____, and its range is _____.
- Suppose the grades turn out all As and Bs.
- Then the range of f is _____, but its codomain is _____.

Operators (general definition)

- An n -ary operator over (or on) the set S is any function from the set of ordered n -tuples of elements of S , to S itself.
- *E.g.*, if $S = \{\mathbf{T}, \mathbf{F}\}$, \neg can be seen as a unary operator, and \wedge, \vee are binary operators on S .
- Another example: \cup and \cap are binary operators on the set of all sets.

Constructing Function Operators

- If \cdot (“dot”) is any operator over B , then we can extend \cdot to also denote an operator over functions $f:A \rightarrow B$.
- *E.g.:* Given any binary operator $\cdot:B \times B \rightarrow B$, and functions $f, g:A \rightarrow B$, we define $(f \cdot g):A \rightarrow B$ to be the function defined by:
 $\forall a \in A, (f \cdot g)(a) = f(a) \cdot g(a)$.

Function Operator Example

- $+, \times$ (“plus”, “times”) are binary operators over \mathbf{R} . (Normal addition & multiplication.)
- Therefore, we can also add and multiply *functions* $f, g: \mathbf{R} \rightarrow \mathbf{R}$:
 - $(f + g): \mathbf{R} \rightarrow \mathbf{R}$, where $(f + g)(x) = f(x) + g(x)$
 - $(f \times g): \mathbf{R} \rightarrow \mathbf{R}$, where $(f \times g)(x) = f(x) \times g(x)$

Function Composition Operator

- For functions $g:A \rightarrow B$ and $f:B \rightarrow C$, there is a special operator called *compose* (“ \circ ”).
 - It composes (creates) a new function out of f and g by applying f to the result of applying g .
 - We say $(f \circ g):A \rightarrow C$, where $(f \circ g)(a) := f(g(a))$.
 - Note $g(a) \in B$, so $f(g(a))$ is defined and $\in C$.
 - Note that \circ (like Cartesian \times , but unlike $+$, \wedge , \cup) is non-commuting. (Generally, $f \circ g \neq g \circ f$.)
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Images of Sets under Functions

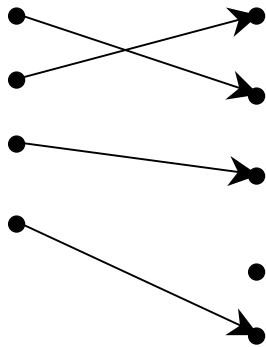
- Given $f:A \rightarrow B$, and $S \subseteq A$,
- The *image* of S under f is simply the set of all images (under f) of the elements of S .
$$f(S) := \{f(s) \mid s \in S\}$$
$$:= \{b \mid \exists s \in S: f(s) = b\}.$$
- Note the range of f can be defined as simply the image (under f) of f 's domain!

One-to-One Functions

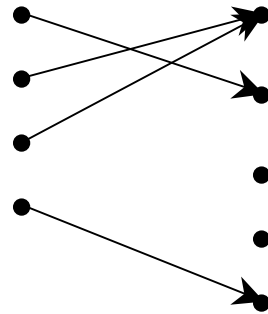
- A function is *one-to-one* (1-1), or *injective*, or *an injection*, iff every element of its range has *only* 1 pre-image.
 - Formally: given $f:A \rightarrow B$,
“ x is injective” $:= (\neg \exists x, y: x \neq y \wedge f(x) = f(y))$.
- Only one element of the domain is mapped to any given one element of the range.
 - Domain & range have same cardinality. What about codomain?
- Memory jogger: Each element of the domain is injected into a different element of the range.
 - Compare “each dose of vaccine is injected into a different patient.”

One-to-One Illustration

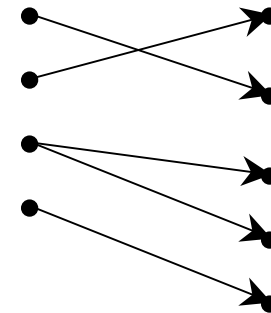
- Bipartite (2-part) graph representations of functions that are (or not) one-to-one:



One-to-one



Not one-to-one



Not even a function!

Sufficient Conditions for 1-1ness

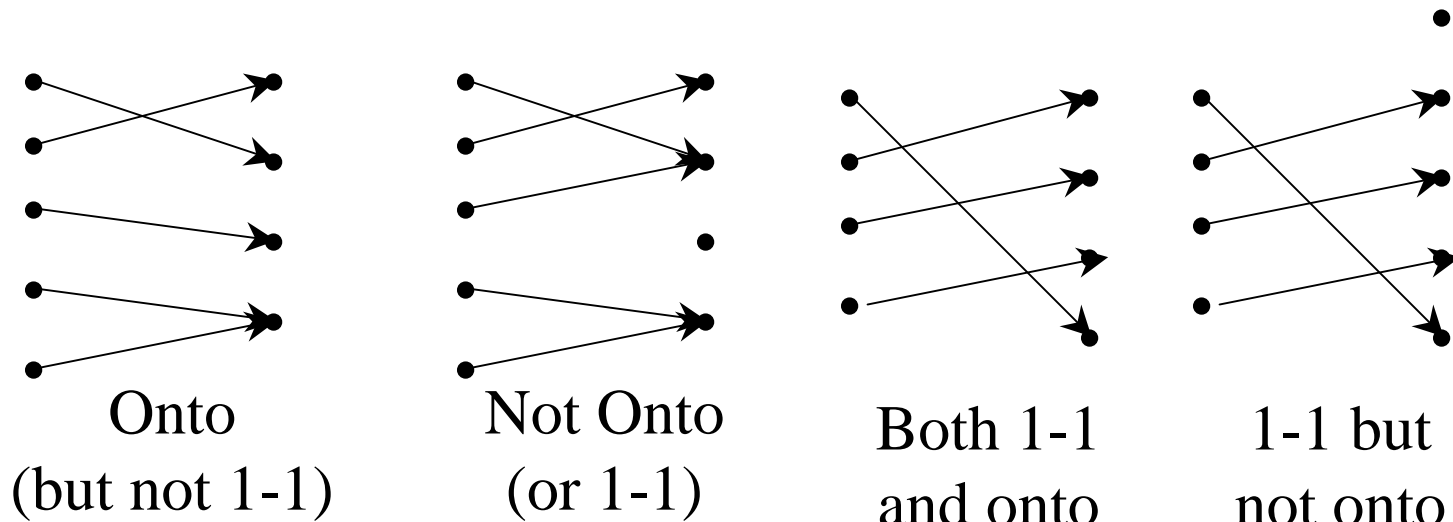
- For functions f over numbers, we say:
 - f is *strictly (or monotonically) increasing* iff $x > y \rightarrow f(x) > f(y)$ for all x, y in domain;
 - f is *strictly (or monotonically) decreasing* iff $x > y \rightarrow f(x) < f(y)$ for all x, y in domain;
- If f is either strictly increasing or strictly decreasing, then f is one-to-one. *E.g. x^3*
 - *Converse is not necessarily true. E.g. $1/x$*

Onto (Surjective) Functions

- A function $f:A\rightarrow B$ is *onto* or *surjective* or a *surjection* iff its range is equal to its codomain ($\forall b\in B, \exists a\in A: f(a)=b$).
- Think: An *onto* function maps the set A onto (over, covering) the *entirety* of the set B , not just over a piece of it.
- E.g., for domain & codomain \mathbf{R} , x^3 is onto, whereas x^2 isn't. (Why not?)

Illustration of Onto

- Some functions that are, or are not, *onto* their codomains:



Bijections

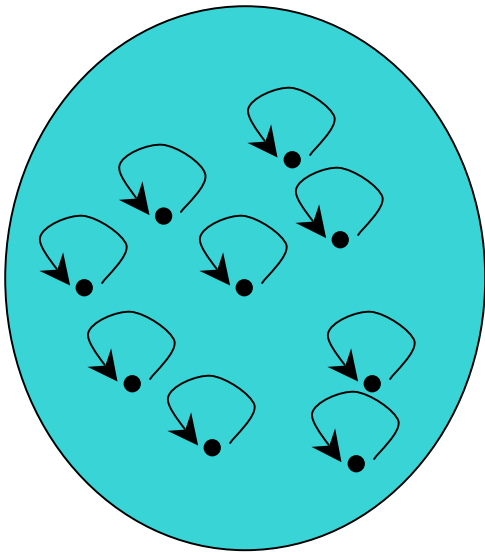
- A function f is said to be a *one-to-one correspondence*, or a *bijection*, or *reversible*, or *invertible*, iff it is both one-to-one and onto.
- For bijections $f:A\rightarrow B$, there exists an *inverse of f* , written $f^{-1}:B\rightarrow A$, which is the unique function such that $f^{-1} \circ f = I_A$
– (where I_A is the identity function on A)

The Identity Function

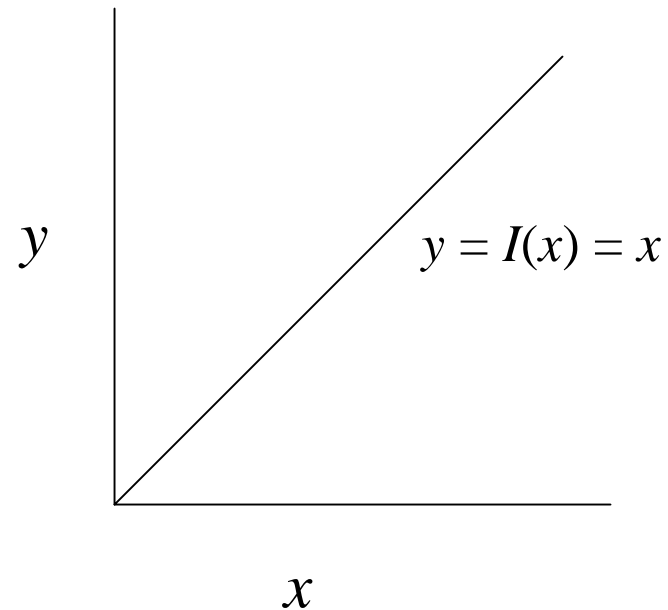
- For any domain A , the *identity function* $I:A \rightarrow A$ (variously written, I_A , $\mathbf{1}$, $\mathbf{1}_A$) is the unique function such that $\forall a \in A: I(a) = a$.
- Some identity functions you've seen:
 - +ing 0, ·ing by 1, ∧ing with \mathbf{T} , ∨ing with \mathbf{F} , ∪ing with \emptyset , ∩ing with U .
- Note that the identity function is always both one-to-one and onto (bijective).

Identity Function Illustrations

- The identity function:



Domain and range

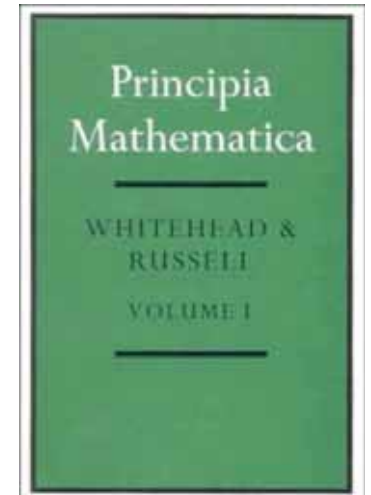


Graphs of Functions

- We can represent a function $f:A \rightarrow B$ as a set of ordered pairs $\{(a, f(a)) \mid a \in A\}$. ← The function's *graph*.
- Note that $\forall a$, there is only 1 pair (a, b) .
 - Later (ch.6): *relations* loosen this restriction.
- For functions over numbers, we can represent an ordered pair (x, y) as a point on a plane.
 - A function is then drawn as a curve (set of points), with only one y for each x .

Aside About Representations

- It is possible to represent any type of discrete structure (propositions, bit-strings, numbers, sets, ordered pairs, functions) in terms of virtually any of the other structures (or some combination thereof).
- Probably none of these structures is truly more fundamental than the others (whatever that would mean). However, strings, logic, and sets are often used as the foundation for all else. *E.g.* in →

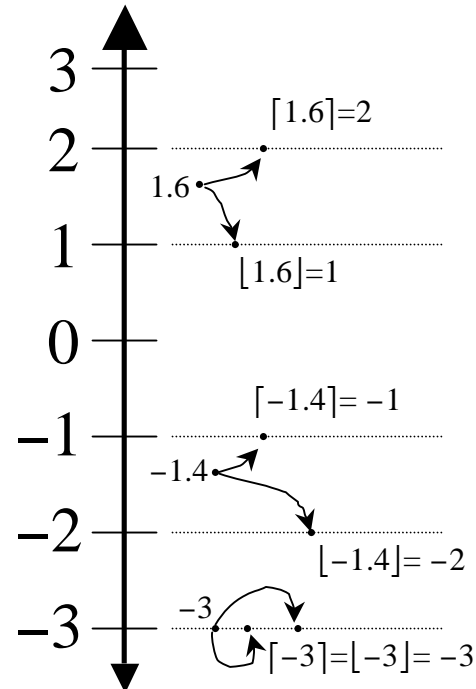


A Couple of Key Functions

- In discrete math, we will frequently use the following two functions over real numbers:
 - The *floor* function $\lfloor \cdot \rfloor : \mathbf{R} \rightarrow \mathbf{Z}$, where $\lfloor x \rfloor$ (“floor of x ”) means the largest (most positive) integer $\leq x$. I.e., $\lfloor x \rfloor := \max(\{i \in \mathbf{Z} \mid i \leq x\})$.
 - The *ceiling* function $\lceil \cdot \rceil : \mathbf{R} \rightarrow \mathbf{Z}$, where $\lceil x \rceil$ (“ceiling of x ”) means the smallest (most negative) integer $\geq x$. $\lceil x \rceil := \min(\{i \in \mathbf{Z} \mid i \geq x\})$

Visualizing Floor & Ceiling

- Real numbers “fall to their floor” or “rise to their ceiling.”
- Note that if $x \notin \mathbf{Z}$,
 $\lfloor -x \rfloor \neq -\lfloor x \rfloor$ &
 $\lceil -x \rceil \neq -\lceil x \rceil$
- Note that if $x \in \mathbf{Z}$,
 $\lfloor x \rfloor = \lceil x \rceil = x$.



Plots with floor/ceiling

- Note that for $f(x)=\lfloor x \rfloor$, the graph of f includes the point $(a, 0)$ for all values of a such that $a \geq 0$ and $a < 1$, but not for the value $a=1$.
- We say that the set of points $(a,0)$ that is in f does not include its *limit* or *boundary* point $(a,1)$.
 - Sets that do not include all of their limit points are generally called *open sets*.
- In a plot, we draw a limit point of a curve using an open dot (circle) if the limit point is not on the curve, and with a closed (solid) dot if it is on the curve.

Plots with floor/ceiling: Example

- Plot of graph of function $f(x) = \lfloor x/3 \rfloor$:

