

Today's topics

- Induction
- Reading: Sections 3.3
- Upcoming
 - More Induction

§3.3: Mathematical Induction

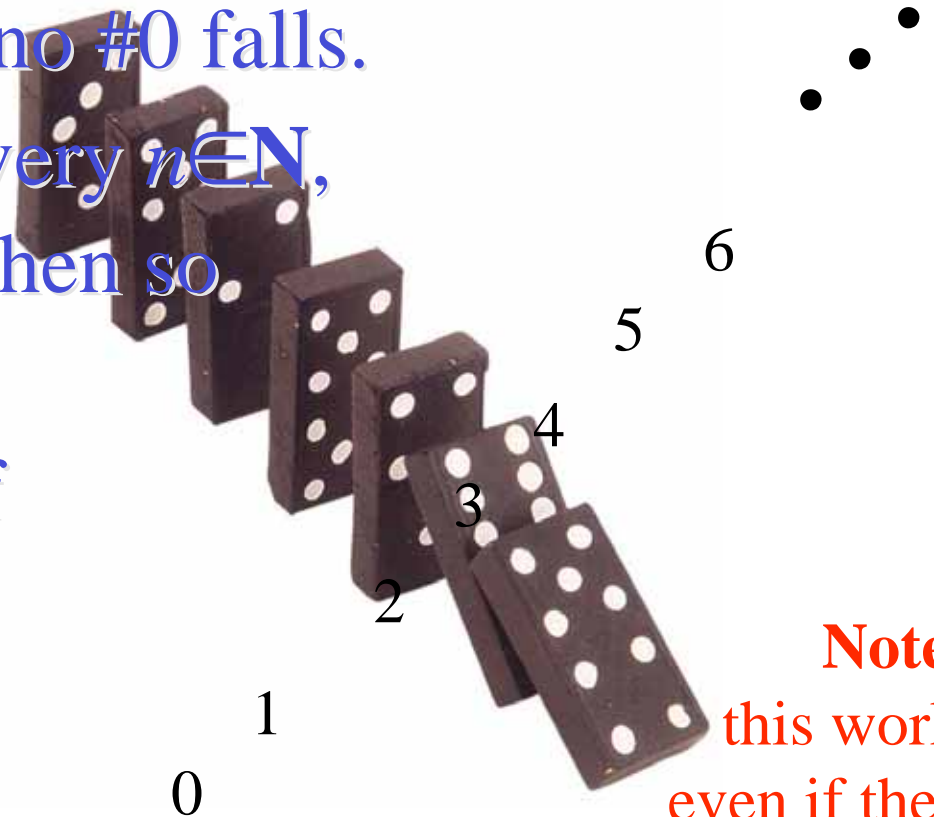
- A powerful, rigorous technique for proving that a predicate $P(n)$ is true for *every* natural number n , no matter how large.
- Essentially a “domino effect” principle.
- Based on a predicate-logic inference rule:

$$\frac{P(0) \quad \forall n \geq 0 (P(n) \rightarrow P(n+1))}{\therefore \forall n \geq 0 P(n)}$$

*“The First Principle
of Mathematical
Induction”*

The “Domino Effect”

- **Premise #1:** Domino #0 falls.
- **Premise #2:** For every $n \in \mathbb{N}$, if domino # n falls, then so does domino # $n+1$.
- **Conclusion:** All of the dominoes fall down!



Note: this works even if there are infinitely many dominoes!

Validity of Induction

Proof that $\forall k \geq 0 P(k)$ is a valid consequent:

Given any $k \geq 0$, the 2nd antecedent

$\forall n \geq 0 (P(n) \rightarrow P(n+1))$ trivially implies that

$\forall n \geq 0 (n < k) \rightarrow (P(n) \rightarrow P(n+1))$, *i.e.*, that $(P(0) \rightarrow P(1)) \wedge (P(1) \rightarrow P(2)) \wedge \dots \wedge (P(k-1) \rightarrow P(k))$.

Repeatedly applying the hypothetical syllogism rule to adjacent implications in this list $k-1$ times then gives us $P(0) \rightarrow P(k)$; which together with $P(0)$ (antecedent #1) and *modus ponens* gives us $P(k)$. Thus $\forall k \geq 0 P(k)$. ■

The Well-Ordering Property

- Another way to prove the validity of the inductive inference rule is by using the *well-ordering property*, which says that:
 - Every non-empty set of non-negative integers has a minimum (smallest) element.
 - $\forall \emptyset \subset S \subseteq \mathbb{N} : \exists m \in S : \forall n \in S : m \leq n$
- This implies that $\{n \mid \neg P(n)\}$ (if non-empty) has a min. element m , but then the assumption that $P(m-1) \rightarrow P((m-1)+1)$ would be contradicted.

Outline of an Inductive Proof

- Let us say we want to prove $\forall n P(n)$...
 - Do the *base case* (or *basis step*): Prove $P(0)$.
 - Do the *inductive step*: Prove $\forall n P(n) \rightarrow P(n+1)$.
 - E.g. you could use a direct proof, as follows:
 - Let $n \in \mathbf{N}$, assume $P(n)$. (*inductive hypothesis*)
 - Now, under this assumption, prove $P(n+1)$.
 - The inductive inference rule then gives us $\forall n P(n)$.

Generalizing Induction

- Rule can also be used to prove $\forall n \geq c P(n)$ for a given constant $c \in \mathbf{Z}$, where maybe $c \neq 0$.
 - In this circumstance, the base case is to prove $P(c)$ rather than $P(0)$, and the inductive step is to prove $\forall n \geq c (P(n) \rightarrow P(n+1))$.
- Induction can also be used to prove $\forall n \geq c P(a_n)$ for any arbitrary series $\{a_n\}$.
- Can reduce these to the form already shown.

Second Principle of Induction

A.k.a. “Strong Induction”

- Characterized by another inference rule:
$$\frac{P(0) \quad \overbrace{P \text{ is true in } \textit{all} \text{ previous cases}}}{\forall n \geq 0: (\forall 0 \leq k \leq n P(k)) \rightarrow P(n+1)}$$
$$\therefore \forall n \geq 0: P(n)$$
- The only difference between this and the 1st principle is that:
 - the inductive step here makes use of the stronger hypothesis that $P(k)$ is true for *all* smaller numbers $k < n+1$, not just for $k=n$.

Induction Example (1st princ.)

- Prove that the sum of the first n odd positive integers is n^2 . That is, prove:

$$\forall n \geq 1: \underbrace{\sum_{i=1}^n (2i - 1)}_{P(n)} = n^2$$

- Proof by induction. $P(n)$
 - Base case: Let $n=1$. The sum of the first 1 odd positive integer is 1 which equals 1^2 .
- (Cont...)

Example cont.

- Inductive step: Prove $\forall n \geq 1: P(n) \rightarrow P(n+1)$.
 - Let $n \geq 1$, assume $P(n)$, and prove $P(n+1)$.

$$\begin{aligned} \sum_{i=1}^{n+1} (2i - 1) &= \left(\sum_{i=1}^n (2i - 1) \right) + (2(n + 1) - 1) \\ &= n^2 + 2n + 1 \\ &= (n + 1)^2 \end{aligned}$$

By inductive hypothesis $P(n)$

Problem

- Show for all natural numbers n
– $(n^3 - n)$ is divisible by 3

Another Induction Example

- Prove that $\forall n > 0, n < 2^n$. Let $P(n) = (n < 2^n)$
 - Base case: $P(1) = (1 < 2^1) = (1 < 2) = \mathbf{T}$.
 - Inductive step: For $n > 0$, prove $P(n) \rightarrow P(n+1)$.
 - Assuming $n < 2^n$, prove $n+1 < 2^{n+1}$.
 - Note $n + 1 < 2^n + 1$ (by inductive hypothesis)
 $< 2^n + 2^n$ (because $1 < 2 = 2 \cdot 2^0 \leq 2 \cdot 2^{n-1} = 2^n$)
 $= 2^{n+1}$
 - So $n + 1 < 2^{n+1}$, and we're done.

Example of Second Principle

- Show that every $n > 1$ can be written as a product $\prod p_i = p_1 p_2 \dots p_s$ of some series of s prime numbers.
 - Let $P(n)$ = “ n has that property”
- **Base case:** $n=2$, let $s=1$, $p_1=2$.
- **Inductive step:** Let $n \geq 2$. Assume $\forall 2 \leq k \leq n: P(k)$.
Consider $n+1$. If it's prime, let $s=1$, $p_1=n+1$.
Else $n+1=ab$, where $1 < a \leq n$ and $1 < b \leq n$.
Then $a=p_1 p_2 \dots p_t$ and $b=q_1 q_2 \dots q_u$. Then we have that $n+1 = p_1 p_2 \dots p_t q_1 q_2 \dots q_u$, a product of $s=t+u$ primes.

Another 2nd Principle Example

- Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps. $P(n)$ = “ n can be...”
- **Base case:** $12=3(4)$, $13=2(4)+1(5)$, $14=1(4)+2(5)$, $15=3(5)$, so $\forall 12 \leq n \leq 15, P(n)$.
- **Inductive step:** Let $n \geq 15$, assume $\forall 12 \leq k \leq n P(k)$. Note $12 \leq n-3 \leq n$, so $P(n-3)$, so add a 4-cent stamp to get postage for $n+1$.

The Method of Infinite Descent

- A way to prove that $P(n)$ is false for all $n \in \mathbf{N}$.
- Sort of a converse to the principle of induction.
- Prove first that $\forall P(n): \exists k < n: P(k)$.
 - Basically, “For every P there is a smaller P .”
- But by the well-ordering property of \mathbf{N} , we know that $\exists P(m) \rightarrow \exists P(n): \forall P(k): n \leq k$.
 - Basically, “If there is a P , there is a smallest P .”
- Note that these are contradictory unless $\neg \exists P(m)$,
 - that is, $\forall m \in \mathbf{N}: \neg P(m)$. There is no P .

Infinite Descent Example

- **Theorem:** $2^{1/2}$ is irrational.
- **Proof:** Suppose $2^{1/2}$ is rational, then $\exists m, n \in \mathbb{Z}^+$: $2^{1/2} = m/n$. Let M, N be the m, n with the least n .

$$\sqrt{2} = \frac{M}{N} \therefore 2 = \frac{M^2}{N^2} \therefore 2N^2 = M^2.$$

$$\frac{2N - M}{M - N} = \frac{(2N - M)N}{(M - N)N} = \frac{2N^2 - MN}{(M - N)N} = \frac{M^2 - MN}{(M - N)N} = \frac{(M - N)M}{(M - N)N} = \frac{M}{N}$$

$$1 < \sqrt{2} < 2 \therefore 1 < \frac{M}{N} < 2 \therefore N < M < 2N \therefore 0 < M - N < N$$

So $\exists k < N, j: 2^{1/2} = j/k$ (let $j = 2N - M, k = M - N$). ■

Problem

- Married couple hosts a party
 - Invites only other married couples
 - At least one person of an invited couple is acquainted to at least the host or the hostess
 - Upon arrival at the party, each person shakes hands with all other guests he/she doesn't know
- Hostess mingles and asks everyone including her husband, “How many hands did you shake?”
 - To her surprise, all responses are different
- How many hands the the host and hostess each shake?