Today's topics

- Relations
  - Kinds of relations
  - n-ary relations
  - Representations of relations
- Reading: Sections 7.1-7.3
- Upcoming
  - Minesweeper

Binary Relations

- Let A, B be any sets. A *binary relation* R from A to B, written (with signature) $R{:}A\times B$, or $R{:}A,B$, is (can be identified with) a subset of the set $A\times B$.
  - *E.g.*, let $<$: $\mathbf{N}\leftrightarrow\mathbf{N} :\equiv \{(n,m) \mid n < m\}$
- The notation $a\ R\ b$ or aRb means that $(a,b)\in R$.
  - *E.g.*, $a < b$ means $(a,b)\in <$
- If aRb we may say "a is related to b (by relation R)",
  - or just "a relates to b (under relation R)".
- A binary relation R corresponds to a predicate function $P_R{:}A\times B\rightarrow\{\mathbf{T},\mathbf{F}\}$ defined over the 2 sets A,B;
  - *e.g.*, predicate "eats" :≡ {(a,b)| organism a eats food b}

Complementary Relations

- Let R:A,B be any binary relation.
- Then, $\not R$:$A\times B$, the *complement* of R, is the binary relation defined by

$$\not R :\equiv \{(a,b) \mid (a,b) \notin R\} = (A\times B) - R$$

- Note this is just $\overline{R}$ if the universe of discourse is $U = A\times B$; thus the name *complement*.
- Note the complement of $\not R$ is R.

Example: $\not< \; = \{(a,b) \mid (a,b) \notin <\} = \{(a,b) \mid \neg a<b\} = \;\geq$

Inverse Relations

- Any binary relation R:$A \times B$ has an *inverse* relation R^{-1}:$B \times A$, defined by
  $$R^{-1} :\equiv \{(b,a) \mid (a,b) \in R\}.$$
  E.g., $<^{-1} = \{(b,a) \mid a<b\} = \{(b,a) \mid b>a\} = >$.
- *E.g.*, if R:People→Foods is defined by
  $a\ R\ b \Leftrightarrow a\ eats\ b$, then:
  $b\ R^{-1}\ a \Leftrightarrow b\ is\ eaten\ by\ a$. (Passive voice.)

Relations on a Set

- A (binary) relation from a set A to itself is called a relation *on* the set A.
- *E.g.*, the "$<$" relation from earlier was defined as a relation *on* the set **N** of natural numbers.
- The (binary) *identity relation* $\mathbf{I}_A$ on a set A is the set $\{(a,a) | a \in A\}$.

Reflexivity

- A relation R on A is *reflexive* if $\forall a \in A,\ aRa$.
  - *E.g.*, the relation $\geq\ :\equiv \{(a,b) \mid a \geq b\}$ is reflexive.
- A relation R is *irreflexive* iff its *complementary* relation R is reflexive.
  - Example: $<$ is irreflexive, because $\geq$ is reflexive.
  - Note "*irreflexive*" does **NOT** mean "*not reflexive*"!
    - For example: the relation "likes" between people is not reflexive, but it is not irreflexive either.
      - Since not everyone likes themselves, but not everyone *dislikes* themselves either!

Symmetry & Antisymmetry

- A binary relation R on A is *symmetric* iff $R = R^{-1}$, that is, if $(a,b)\in R \leftrightarrow (b,a)\in R$.
  - *E.g.*, **=** (equality) is symmetric. < is not.
  - "is married to" is symmetric, "likes" is not.
- A binary relation R is *antisymmetric* if $\forall a\neq b, (a,b)\in R \rightarrow (b,a)\notin R$.
  - **Examples:** < is antisymmetric, "likes" is not.
  - **Exercise:** prove this definition of antisymmetric is equivalent to the statement that $R\cap R^{-1} \subseteq \mathbf{I}_A$.

Transitivity

- A relation R is *transitive* iff (for all a,b,c) $(a,b)\in R \land (b,c)\in R \rightarrow (a,c)\in R$.
- A relation is *intransitive* iff it is not transitive.
- Some examples:
  - "is an ancestor of" is transitive.
  - "likes" between people is intransitive.
  - "is located within 1 mile of" is… ?

Totality

- A relation $R{:}A\times B$ is *total* if for every $a\in A$, there is at least one $b\in B$ such that $(a,b)\in R$.
- If R is not total, then it is called *strictly partial*.
- A *partial relation* is a relation that *might* be strictly partial. (Or, it might be total.)
  - In other words, *all* relations are considered "partial."

Functionality

- A relation $R:A\times B$ is *functional* if, for any $a\in A$, there is *at most 1* $b\in B$ such that $(a,b)\in R$.
  - "R is functional" $\Leftrightarrow$ $\forall a\in A: \neg\exists b_1\neq b_2\in B: aRb_1 \wedge aRb_2$.
  - Iff R is functional, then it corresponds to a partial function $R:A\rightarrow B$
    - where $R(a)=b \Leftrightarrow aRb$; *e.g.*
      - *E.g.*, The relation $aRb :\equiv$ "$a + b = 0$" yields the function $-(a) = b$.
- R is *antifunctional* if its inverse relation R^{-1} is functional.
  - Note: A functional relation (partial function) that is also antifunctional is an *invertible* partial function.
- R is a *total function* $R:A\rightarrow B$ if it is both functional and total, that is, for any $a\in A$, there is *exactly* 1 b such that $(a,b)\in R$. *I.e.*, $\forall a\in A: \neg\exists! b: aRb$.
  - If R is functional but not total, then it is a *strictly partial function*.
  - **Exercise:** Show that iff R is functional and antifunctional, and both it and its inverse are total, then it is a bijective function.

Lambda Notation

- The *lambda calculus* is a formal mathematical language for defining and operating on functions.
  - It is the mathematical foundation of a number of functional (recursive function-based) programming languages, such as LISP and ML.
- It is based on *lambda notation*, in which "λa: $f(a)$" is a way to denote the function *f without ever assigning it a specific symbol*.
  - *E.g.*, $(\lambda x.\ 3x^2+2x)$ is a name for the function f:**R**→**R** where $f(x)=3x^2+2x$.
- Lambda notation and the "such that" operator "∍" can also be used to compose an expression for the function that corresponds to any given functional relation.
  - Let $R{:}A\times B$ be any functional relation on A,B.
  - Then the expression $(\lambda a\text{: } b \ni aRb)$ denotes the function $f{:}A\rightarrow B$ where $f(a) = b$ iff aRb.
    - E.g., If I write: $f :\equiv (\lambda a\text{: } b \ni a+b = 0)$, this is one way of defining the function $f(a)=-a$.

Composite Relations

- Let $R:A\times B$, and $S:B\times C$. Then the *composite* $S\circ R$ of R and S is defined as:

  $$S\circ R = \{(a,c) \mid \exists b: aRb \wedge bSc\}$$

- Note that function composition $f\circ g$ is an example.
- **Exer.**: Prove that $R:A\times A$ is transitive iff $R\circ R = R$.
- The n^{th} power R^n of a relation R on a set A can be defined recursively by:

  $$R^0 :\equiv \mathbf{I}_A\,; \qquad R^{n+1} :\equiv R^n\circ R \quad \text{for all } n\geq 0.$$

  - Negative powers of R can also be defined if desired, by $R^{-n} :\equiv (R^{-1})^n$.

§7.2: *n*-ary Relations

- An *n*-ary relation *R* on sets $A_1,\ldots,A_n$, written (with signature) $R{:}A_1\times\ldots\times A_n$ or $R{:}A_1,\ldots,A_n$, is simply a subset
$$R \subseteq A_1\times \ldots \times A_n.$$
- The sets A_i are called the *domains* of *R*.
- The *degree* of *R* is *n*.
- *R* is *functional in the domain* A_i if it contains at most one *n*-tuple $(\ldots, a_i ,\ldots)$ for any value a_i within domain A_i.

Relational Databases

- A *relational database* is essentially just an n-ary relation R.
- A domain A_i is a *primary key* for the database if the relation R is functional in A_i.
- A *composite key* for the database is a set of domains $\{A_i, A_j, \ldots\}$ such that R contains at most 1 n-tuple $(\ldots,a_i,\ldots,a_j,\ldots)$ for each composite value $(a_i, a_j,\ldots)\in A_i\times A_j\times\ldots$

Selection Operators

- Let A be any n-ary domain $A = A_1 \times \ldots \times A_n$, and let $C: A \rightarrow \{\mathbf{T}, \mathbf{F}\}$ be any *condition* (predicate) on elements (n-tuples) of A.
- Then, the *selection operator* s_C is the operator that maps any (n-ary) relation R on A to the n-ary relation of all n-tuples from R that satisfy C.
  - *I.e.*, $\forall R \subseteq A,\ s_C(R) = \{a \in R \mid s_C(a) = \mathbf{T}\}$

Selection Operator Example

- Suppose we have a domain
  A = StudentName × Standing × SocSecNos
- Suppose we define a certain condition on A,
  $UpperLevel(name, standing, ssn) :\equiv$
  $[(standing = \text{junior}) \vee (standing = \text{senior})]$
- Then, $s_{UpperLevel}$ is the selection operator that takes any relation R on A (database of students) and produces a relation consisting of *just* the upper-level classes (juniors and seniors).

Projection Operators

- Let $A = A_1 \times \ldots \times A_n$ be any *n*-ary domain, and let $\{i_k\}=(i_1,\ldots,i_m)$ be a sequence of indices all falling in the range 1 to *n*,
  - That is, where $1 \leq i_k \leq n$ for all $1 \leq k \leq m$.
- Then the *projection operator* on *n*-tuples

$$P_{\{i_k\}} : A \rightarrow A_{i_1} \times \ldots \times A_{i_m}$$

is defined by:

$$P_{\{i_k\}}(a_1,\ldots,a_n) = (a_{i_1},\ldots,a_{i_m})$$

Projection Example

- Suppose we have a ternary (3-ary) domain $Cars = Model \times Year \times Color$. (note n=3).
- Consider the index sequence $\{i_k\} = 1,3$. (m=2)
- Then the projection $P_{\{i_k\}}$ simply maps each tuple $(a_1,a_2,a_3) = (model, year, color)$ to its image:

$$(a_{i_1}, a_{i_2}) = (a_1, a_3) = (model, color)$$

- This operator can be usefully applied to a whole relation $R \subseteq Cars$ (a database of cars) to obtain a list of the model/color combinations available.

Join Operator

- Puts two relations together to form a sort of combined relation.
- If the tuple (A,B) appears in R_1, and the tuple (B,C) appears in R_2, then the tuple (A,B,C) appears in the join $J(R_1,R_2)$.
  - A, B, and C here can also be sequences of elements (across multiple fields), not just single elements.

Join Example

- Suppose R_1 is a teaching assignment table, relating *Professors* to *Courses*.
- Suppose R_2 is a room assignment table relating *Courses* to *Rooms*,*Times*.
- Then $J(R_1,R_2)$ is like your class schedule, listing *(professor,course,room,time)*.

§7.3: Representing Relations

- Some ways to represent *n*-ary relations:
  - With an explicit list or table of its tuples.
  - With a function from the domain to **{T,F}**.
    - Or with an algorithm for computing this function.
- Some special ways to represent binary relations:
  - With a zero-one matrix.
  - With a directed graph.

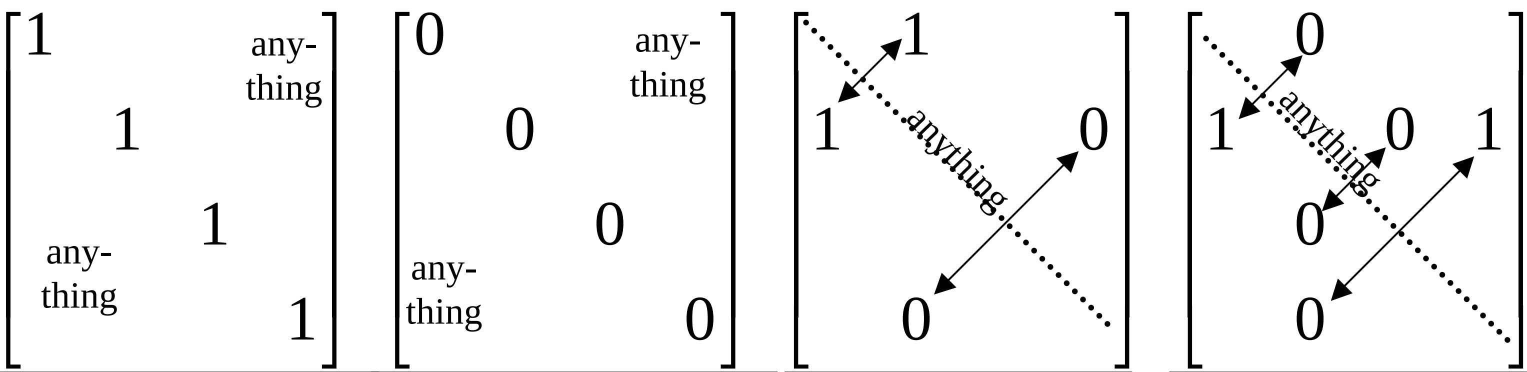
Using Zero-One Matrices

- To represent a binary relation $R:A\times B$ by an $|A|\times|B|$ 0-1 matrix $\mathbf{M}_R = [m_{ij}]$, let $m_{ij} = 1$ iff $(a_i,b_j)\in R$.
- *E.g.*, Suppose Joe likes Susan and Mary, Fred likes Mary, and Mark likes Sally.
- Then the 0-1 matrix representation of the relation Likes:Boys×Girls relation is:

	Susan	Mary	Sally
Joe	1	1	0
Fred	0	1	0
Mark	0	0	1

Zero-One Reflexive, Symmetric

- Terms: *Reflexive*, *non-reflexive*, *irreflexive*, *symmetric*, *asymmetric*, *and antisymmetric*.
  - These relation characteristics are very easy to recognize by inspection of the zero-one matrix.



Reflexive: all 1's on diagonal	*Irreflexive*: all 0's on diagonal	*Symmetric*: all identical across diagonal	*Antisymmetric*: all 1's are across from 0's

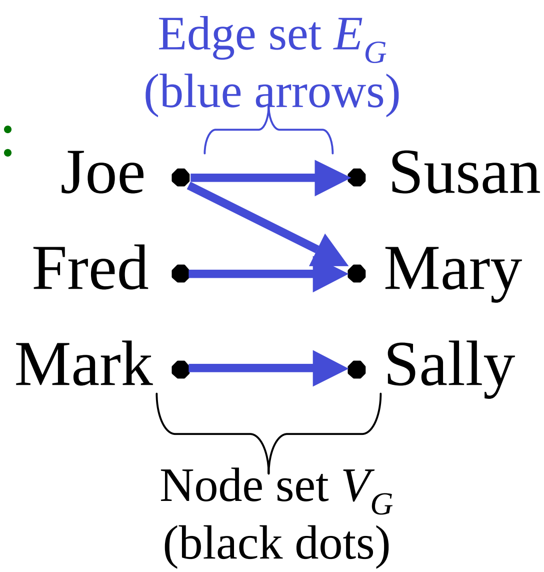
Using Directed Graphs

- A *directed graph* or *digraph* $G=(V_G,E_G)$ is a set V_G of *vertices (nodes)* with a set $E_G \subseteq V_G \times V_G$ of *edges (arcs,links)*. Visually represented using dots for nodes, and arrows for edges. Notice that a relation $R{:}A \times B$ can be represented as a graph $G_R=(V_G=A \cup B,\ E_G=R)$.

Matrix representation $\mathbf{M}_R$:

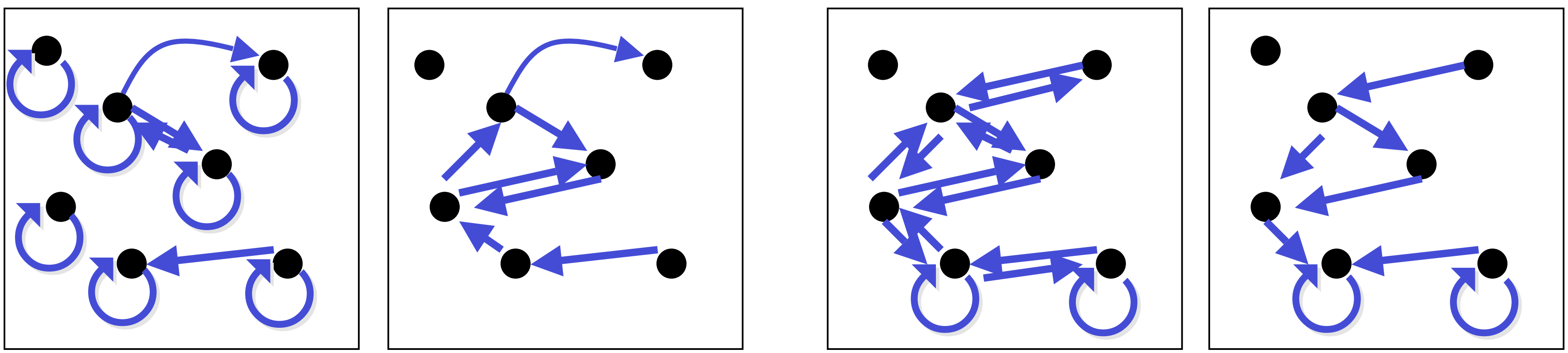
	Susan	Mary	Sally
Joe	1	1	0
Fred	0	1	0
Mark	0	0	1

Graph rep. G_R:



Digraph Reflexive, Symmetric

It is extremely easy to recognize the reflexive/irreflexive/ symmetric/antisymmetric properties by graph inspection.



Reflexive: Every node has a self-loop	Irreflexive: No node links to itself	Symmetric: Every link is bidirectional	Antisymmetric: No link is bidirectional

These are asymmetric & non-antisymmetric

These are non-reflexive & non-irreflexive