Today’s topics

- Functions
  - Notations and terms
  - One-to-One vs. Onto
  - Floor, ceiling, and identity
- Reading: Sections 1.8
- Upcoming
  - Algorithms

On to section 1.8... Functions

- From calculus, you are familiar with the concept of a real-valued function \( f \), which assigns to each number \( x \in \mathbb{R} \) a particular value \( y = f(x) \), where \( y \in \mathbb{R} \).
- But, the notion of a function can also be naturally generalized to the concept of assigning elements of any set to elements of any set. (Also known as a map.)

Function: Formal Definition

- For any sets \( A, B \), we say that a function \( f \) from (or “mapping”) \( A \) to \( B \) (\( f: A \rightarrow B \)) is a particular assignment of exactly one element \( f(x) \in B \) to each element \( x \in A \).
- Some further generalizations of this idea:
  - A partial (non-total) function \( f \) assigns zero or one elements of \( B \) to each element \( x \in A \).
  - Functions of \( n \) arguments; relations (ch. 6).

Graphical Representations

- Functions can be represented graphically in several ways:
  - Graphs
  - Bipartite Graph
  - Like Venn diagrams
  - Plots
**Functions We’ve Seen So Far**

- A *proposition* can be viewed as a function from “situations” to truth values \( \{T, F\} \)
  - A logic system called *situation theory*.
  - \( p = “It is raining.”; s = our situation here, now \)
  - \( p(s) \in \{T, F\} \).
- A *propositional operator* can be viewed as a function from ordered pairs of truth values to truth values: e.g., \( \lor((F,T)) = T \).
  
  Another example: \( \rightarrow((T,F)) = F \).

**A Neat Trick**

- Sometimes we write \( Y^X \) to denote the set \( F \) of all possible functions \( f : X \rightarrow Y \).
- This notation is especially appropriate, because for finite \( X, Y \), we have \( |F| = |Y|^{|X|} \).
- If we use representations \( F = 0, T = 1, 2 := \{0,1\} = \{F,T\} \), then a subset \( T \subseteq S \) is just a function from \( S \) to \( 2 \), so the power set of \( S \) (set of all such fns.) is \( 2^S \) in this notation.

**Some Function Terminology**

- If it is written that \( f : A \rightarrow B \), and \( f(a) = b \) (where \( a \in A \& b \in B \)), then we say:
  - \( A \) is the *domain* of \( f \).
  - \( B \) is the *codomain* of \( f \).
  - \( b \) is the *image* of \( a \) under \( f \).
  - \( a \) is a *pre-image* of \( b \) under \( f \).
  - We also say the *signature* of \( f \) is \( A \rightarrow B \).
  - In general, \( b \) may have more than 1 pre-image.
  - The *range* \( R \subseteq B \) of \( f \) is \( R = \{b \mid \exists a f(a) = b\} \).

**Range versus Codomain**

- The range of a function might *not* be its whole codomain.
- The codomain is the set that the function is *declared* to map all domain values into.
- The range is the *particular* set of values in the codomain that the function *actually* maps elements of the domain to.
Range vs. Codomain - Example

• Suppose I declare to you that: “\( f \) is a function mapping students in this class to the set of grades \( \{A,B,C,D,E\} \).”
• At this point, you know \( f \)’s codomain is: __________, and its range is ________.
• Suppose the grades turn out all As and Bs.
• Then the range of \( f \) is __________, but its codomain is __________________.

Operators (general definition)

• An \( n \)-ary operator over (or on) the set \( S \) is any function from the set of ordered \( n \)-tuples of elements of \( S \), to \( S \) itself.
• E.g., if \( S=\{\text{T,F}\} \), \( \lnot \) can be seen as a unary operator, and \( \land,\lor \) are binary operators on \( S \).
• Another example: \( \cup \) and \( \cap \) are binary operators on the set of all sets.

Constructing Function Operators

• If \( \bullet \) (“dot”) is any operator over \( B \), then we can extend \( \bullet \) to also denote an operator over functions \( f:A\rightarrow B \).
• E.g.: Given any binary operator \( \bullet:B\times B\rightarrow B \), and functions \( f,g:A\rightarrow B \), we define \( (f\bullet g):A\rightarrow B \) to be the function defined by: \( \forall a\in A, \ (f\bullet g)(a) = f(a)\bullet g(a) \).

Function Operator Example

• \( +,\times \) (“plus”, “times”) are binary operators over \( \mathbb{R} \). (Normal addition & multiplication.)
• Therefore, we can also add and multiply functions \( f,g:\mathbb{R}\rightarrow\mathbb{R} \):
  – \( f+g:\mathbb{R}\rightarrow\mathbb{R}, \) where \( (f+g)(x) = f(x) + g(x) \)
  – \( f\times g:\mathbb{R}\rightarrow\mathbb{R}, \) where \( (f\times g)(x) = f(x) \times g(x) \)
Function Composition Operator

- For functions $g:A \rightarrow B$ and $f:B \rightarrow C$, there is a special operator called *compose* ("\(\circ\)").
  - It *composes* (creates) a new function out of $f$ and $g$ by applying $f$ to the result of applying $g$.
  - We say $(f \circ g):A \rightarrow C$, where $(f \circ g)(a) := f(g(a))$.
  - Note $g(a) \in B$, so $f(g(a))$ is defined and $\in C$.
  - Note that $\circ$ (like Cartesian $\times$, but unlike $+, \land, \cup$) is non-commuting. (Generally, $f \circ g \neq g \circ f$.)

Images of Sets under Functions

- Given $f:A \rightarrow B$, and $S \subseteq A$,
- The *image* of $S$ under $f$ is simply the set of all images (under $f$) of the elements of $S$.
  
  $f(S) := \{ f(s) \mid s \in S \}$
  
  $:= \{ b \mid \exists s \in S: f(s) = b \}$.  
- Note the range of $f$ can be defined as simply the image (under $f$) of $f$’s domain!

One-to-One Functions

- A function is *one-to-one* (1-1), or *injective*, or an *injection*, iff every element of its range has only 1 pre-image.
  - Formally: given $f:A \rightarrow B$,
    
    "$x$ is injective" $:= (\neg \exists x, y: x \neq y \land f(x) = f(y))$.
  - Only one element of the domain is mapped to any given one element of the range.
    - Domain & range have same cardinality. What about codomain?
  - Memory jogger: Each element of the domain is injected into a different element of the range.
    - Compare “each dose of vaccine is injected into a different patient.”

One-to-One Illustration

- Bipartite (2-part) graph representations of functions that are (or not) one-to-one:
  - **One-to-one**
  - **Not one-to-one**
  - **Not even a function!**
Sufficient Conditions for 1-1ness

- For functions $f$ over numbers, we say:
  - $f$ is strictly (or monotonically) increasing iff $x>y \rightarrow f(x) > f(y)$ for all $x, y$ in domain;
  - $f$ is strictly (or monotonically) decreasing iff $x>y \rightarrow f(x) < f(y)$ for all $x, y$ in domain;
- If $f$ is either strictly increasing or strictly decreasing, then $f$ is one-to-one. E.g. $x^3$
  - Converse is not necessarily true. E.g. $1/x$

Onto (Surjective) Functions

- A function $f:A \rightarrow B$ is onto or surjective or a surjection iff its range is equal to its codomain ($\forall b \in B, \exists a \in A: f(a) = b$).
- Think: An onto function maps the set $A$ onto (over, covering) the entirety of the set $B$, not just over a piece of it.
- E.g., for domain & codomain $\mathbb{R}$, $x^3$ is onto, whereas $x^2$ isn’t. (Why not?)

Illustration of Onto

- Some functions that are, or are not, onto their codomains:

| Onto (but not 1-1) | Not Onto (or 1-1) | Both 1-1 and onto | 1-1 but not onto |

Bijections

- A function $f$ is said to be a one-to-one correspondence, or a bijection, or reversible, or invertible, iff it is both one-to-one and onto.
- For bijections $f:A \rightarrow B$, there exists an inverse of $f$, written $f^{-1}:B \rightarrow A$, which is the unique function such that $f^{-1} \circ f = I_A$
  - (where $I_A$ is the identity function on $A$)
The Identity Function

- For any domain $A$, the identity function $I:A \rightarrow A$ (variously written, $I_A$, $1$, $I_A$) is the unique function such that $\forall a \in A: I(a) = a$.
- Some identity functions you’ve seen:
  - $+_0$, $\cdot 0$, $\cdot 1$
  - $\lor$, $\land$ with $\top$, $\land$ with $\bot$
- Note that the identity function is always both one-to-one and onto (bijective).

Identity Function Illustrations

- The identity function:

Graphs of Functions

- We can represent a function $f:A \rightarrow B$ as a set of ordered pairs $\{(a, f(a)) \mid a \in A\}$. ← The function’s graph.
- Note that $\forall a$, there is only 1 pair $(a, b)$.
  - Later (ch.6): relations loosen this restriction.
- For functions over numbers, we can represent an ordered pair $(x, y)$ as a point on a plane.
  - A function is then drawn as a curve (set of points), with only one $y$ for each $x$.

Aside About Representations

- It is possible to represent any type of discrete structure (propositions, bit-strings, numbers, sets, ordered pairs, functions) in terms of virtually any of the other structures (or some combination thereof).
- Probably none of these structures is truly more fundamental than the others (whatever that would mean). However, strings, logic, and sets are often used as the foundation for all else. E.g. in Principia Mathematica.
A Couple of Key Functions

- In discrete math, we will frequently use the following two functions over real numbers:
  - The floor function $\lfloor \cdot \rceil : \mathbb{R} \rightarrow \mathbb{Z}$, where $\lfloor x \rceil$ ("floor of $x$") means the largest (most positive) integer $\leq x$. I.e., $\lfloor x \rceil : = \max(\{i \in \mathbb{Z} | i \leq x\})$.
  - The ceiling function $\lceil \cdot \rceil : \mathbb{R} \rightarrow \mathbb{Z}$, where $\lceil x \rceil$ ("ceiling of $x$") means the smallest (most negative) integer $\geq x$. $\lceil x \rceil : = \min(\{i \in \mathbb{Z} | i \geq x\})$.

Visualizing Floor & Ceiling

- Real numbers “fall to their floor” or “rise to their ceiling.”
- Note that if $x \notin \mathbb{Z}$, $\lfloor -x \rceil \neq -\lceil x \rceil$ & $\lceil -x \rceil \neq -\lfloor x \rceil$.
- Note that if $x \in \mathbb{Z}$, $\lfloor x \rceil = \lfloor x \rceil = x$.

Plots with floor/ceiling

- Note that for $f(x) = \lfloor x \rceil$, the graph of $f$ includes the point $(a, 0)$ for all values of $a$ such that $a \neq 0$ and $a < 1$, but not for the value $a = 1$.
- We say that the set of points $(a, 0)$ that is in $f$ does not include its limit or boundary point $(a, 1)$.
  - Sets that do not include all of their limit points are generally called open sets.
- In a plot, we draw a limit point of a curve using an open dot (circle) if the limit point is not on the curve, and with a closed (solid) dot if it is on the curve.

Plots with floor/ceiling: Example

- Plot of graph of function $f(x) = \lfloor x/3 \rceil$:
  - Set of points $(x, f(x))$.