§3.4: Recursive Definitions

- In induction, we prove all members of an infinite set satisfy some predicate $P$ by:
  - proving the truth of the predicate for larger members in terms of that of smaller members.
- In recursive definitions, we similarly define a function, a predicate, a set, or a more complex structure over an infinite domain (universe of discourse) by:
  - defining the function, predicate value, set membership, or structure of larger elements in terms of those of smaller ones.
- In structural induction, we inductively prove properties of recursively-defined objects in a way that parallels the objects’ own recursive definitions.

Recursively Defined Functions

- Simplest case: One way to define a function $f: \mathbb{N} \rightarrow S$ (for any set $S$) or series $a_n = f(n)$ is to:
  - Define $f(0)$.
  - For $n>0$, define $f(n)$ in terms of $f(0), \ldots, f(n-1)$.
- E.g.: Define the series $a_n := 2^n$ recursively:
  - Let $a_0 := 1$.
  - For $n>0$, let $a_n := 2a_{n-1}$.

Recursion

- **Recursion** is the general term for the practice of defining an object in terms of itself
  - or of part of itself
  - This may seem circular, but it isn’t necessarily.
- An inductive proof establishes the truth of $P(n+1)$ **recursively** in terms of $P(n)$.
- There are also recursive algorithms, definitions, functions, sequences, sets, and other structures.

Today’s topics

- Recursion
  - Recursively defined functions
  - Recursively defined sets
  - Structural Induction
- Reading: Sections 3.4
- Upcoming
  - Counting
Another Example

• Suppose we define $f(n)$ for all $n \in \mathbb{N}$ recursively by:
  – Let $f(0)=3$
  – For all $n \in \mathbb{N}$, let $f(n+1)=2f(n)+3$
• What are the values of the following?
  – $f(1) = 9$  $f(2) = 21$  $f(3) = 45$  $f(4) = 93$

Recursive definition of Factorial

• Give an inductive (recursive) definition of the factorial function,
  $$F(n) := n! := \prod_{1 \leq i \leq n} i = 1 \cdot 2 \cdots n.$$  
  – Base case: $F(0) := 1$
  – Recursive part: $F(n) := n \cdot F(n-1)$.
    • $F(1) = 1$
    • $F(2) = 2$
    • $F(3) = 6$

More Easy Examples

• Write down recursive definitions for:
  $i+n$ ($i$ integer, $n$ natural) using only $s(i) = i+1$.
  $a \cdot n$ ($a$ real, $n$ natural) using only addition
  $a^n$ ($a$ real, $n$ natural) using only multiplication
  $\sum_{0 \leq i \leq n} a_i$ (for an arbitrary series of numbers $\{a_i\}$)
  $\prod_{0 \leq i \leq n} a_i$ (for an arbitrary series of numbers $\{a_i\}$)
  $\bigcap_{0 \leq i \leq n} S_i$ (for an arbitrary series of sets $\{S_i\}$)

The Fibonacci Series

• The Fibonacci series $f_{n \geq 0}$ is a famous series defined by:
  $$f_0 := 0, \quad f_1 := 1, \quad f_{n \geq 2} := f_{n-1} + f_{n-2}$$

Leonardo Fibonacci
1170-1250
Inductive Proof about Fib. series

• **Theorem:** \(f_n < 2^n\). \(\Longleftrightarrow\) Implicitly for all \(n \in \mathbb{N}\)
• **Proof:** By induction.

Base cases: \(f_0 = 0 < 2^0 = 1\) \(f_1 = 1 < 2^1 = 2\)

Inductive step: Use 2nd principle of induction (strong induction). Assume \(\forall k \leq n, f_k < 2^k\). Then \(f_n = f_{n-1} + f_{n-2}\) is
\[< 2^{n-1} + 2^{n-2} < 2^{n-1} + 2^{n-1} = 2^n.\]

A lower bound on Fibonacci series

• **Theorem.** For all integers \(n \geq 3, f_n > a^{n-2}\), where \(\alpha = (1+5^{1/2})/2 \approx 1.61803\).
• **Proof.** (Using strong induction.)
  - Let \(P(n) = (f_n > a^{n-2})\).
  - **Base cases:** For \(n=3\), note that \(a < 2 = f_3\). For \(n=4, a^2 = (1+2.5^{1/2})/4 = (3+5^{1/2})/2 \approx 2.61803 < 3 = f_4\).
  - **Inductive step:** For \(k \geq 4\), assume \(P(j)\) for \(3 \leq j \leq k\), prove \(P(k+1)\). Note \(a^3 = \alpha + 1\). Thus, \(a^k = a^{k-1} + a^{k-2}\). By inductive hypothesis, \(f_{k-1} > a^{k-2}\) and \(f_k > a^{k-2}\). So, \(f_{k+1} = f_k + f_{k-1} > a^{k-2} + a^{k-3} = a^{k-1}\). Thus \(P(k+1)\).

Lamé’s Theorem

• **Theorem:** \(\forall a,b \in \mathbb{N}, a \geq b > 0\), the number of steps in Euclid’s algorithm to find \(\text{gcd}(a, b)\) is \(\leq 5k\), where \(k = \lfloor \log_{10} b \rfloor + 1\) is the number of decimal digits in \(b\).
  - Thus, Euclid’s algorithm is linear-time in the number of digits in \(b\).
• **Proof:**
  - Uses the Fibonacci sequence!
  - See next 2 slides.

Proof of Lamé’s Theorem

• Consider the sequence of division-algorithm equations used in Euclid’s alg.:
  \[
  r_0 = r_1 q_1 + r_2 \\
  r_1 = r_2 q_2 + r_3 \\
  \vdots \\
  r_{n-2} = r_{n-1} q_{n-1} + r_n \\
  r_{n-1} = r_n q_n + r_{n+1} \\
  \]
  with \(0 \leq r_2 < r_1\), \(0 \leq r_3 < r_2\), etc.

  \[
  r_{n-2} = r_{n-1} q_{n-1} + r_n \\
  r_{n-1} = r_n q_n + r_{n+1}
  \]
  with \(r_{n+1} = 0\) (terminate)

  Where \(a = r_0\), \(b = r_1\), and \(\text{gcd}(a,b) = r_n\).

  The number of divisions (iterations) is \(n\).
The Set of All Strings

- Given an alphabet $\Sigma$, the set $\Sigma^*$ of all strings over $\Sigma$ can be recursively defined by:
  - $\varepsilon \in \Sigma^*$ (the empty string)
  - $w \in \Sigma^* \land x \in \Sigma \rightarrow wx \in \Sigma^*$

- **Exercise**: Prove that this definition is equivalent to our old one: $\Sigma^* = \bigcup_{n \in \mathbb{N}} \Sigma^n$.
Rooted Trees

- Trees will be covered in CompSci 130.
  - Briefly, a tree is a graph in which there is exactly one
    undirected path between each pair of nodes.
  - An undirected graph can be represented as a set of
    unordered pairs (called arcs) of objects called nodes.
- Definition of the set of rooted trees:
  - Any single node \( r \) is a rooted tree.
  - If \( T_1, \ldots, T_n \) are disjoint rooted trees with respective
    roots \( r_1, \ldots, r_n \), and \( r \) is a node not in any of the \( T_i \)'s,
    then another rooted tree is \( \{r, r_1\}, \ldots, \{r, r_n\} \cup T_1 \cup \cdots \cup T_n \).

Extended Binary Trees

- A special case of rooted trees.
- Recursive definition of EBTs:
  - The empty set \( \emptyset \) is an extended binary tree.
  - If \( T_1, T_2 \) are disjoint EBTs, then \( e_1 \cup e_2 \cup T_1 \cup T_2 \)
    is an EBT, where \( e_1 = \emptyset \) if \( T_1 = \emptyset \), and \( e_1 = \{(r, r_1)\} \) if \( T_1 \neq \emptyset \)
    and has root \( r_1 \), and similarly for \( e_2 \).

Illustrating Rooted Tree Def’n.

- How rooted trees can be combined to form a new rooted tree…
  \[ T_1 \quad T_2 \quad \cdots \quad T_n \]

Full Binary Trees

- A special case of extended binary trees.
- Recursive definition of FBTs:
  - A single node \( r \) is a full binary tree.
    - Note this is different from the EBT base case.
  - If \( T_1, T_2 \) are disjoint FBTs, then \( e_1 \cup e_2 \cup T_1 \cup T_2 \), where \( e_1 = \emptyset \) if \( T_1 \)
    is \( \emptyset \), and \( e_1 = \{(r, r_1)\} \) if \( T_1 \neq \emptyset \) and has root \( r_1 \), and similarly for \( e_2 \).
    - Note this is the same as the EBT recursive case!
      - Can simplify it to “If \( T_1, T_2 \) are disjoint FBTs with roots \( r_1 \) and \( r_2 \), then
        \( \{(r, r_1), (r, r_2)\} \cup T_1 \cup T_2 \) is an FBT.”
Structural Induction

• Proving something about a recursively defined object using an inductive proof whose structure mirrors the object’s definition.

• **Example problem:** Let $3 \in S$, and let $x+y \in S$ if $x, y \in S$. Show that $S = \{n \in \mathbb{Z}^+ | (3|n)\}$ (the set of positive multiples of 3).

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Example continued

• Let $3 \in S$, and let $x+y \in S$ if $x, y \in S$. Let $A = \{n \in \mathbb{Z}^+ | (3|n)\}$.

• **Theorem:** $A = S$. **Proof:** We show that $A \subseteq S$ and $S \subseteq A$.
  
  – To show $A \subseteq S$, show $[n \in \mathbb{Z}^+ \wedge (3|n)] \rightarrow n \in S$.

  • **Inductive proof.** Let $P(n) : n \in S$. Induction over positive multiples of 3. Base case: $n=3$, thus $3 \in S$ by defn. of $S$. Inductive step: Given $P(n)$, prove $P(n+3)$. By inductive hyp., $n \in S$, and $3 \in S$, so by defn of $S$, $n+3 \in S$.

  – To show $S \subseteq A$: let $n \in S$, show $n \in A$.

  • **Structural inductive proof.** Let $P(n) : n \in A$. Two cases: $n=3$ (base case), which is in $A$, or $n=x+y$ (recursive step). We know $x$ and $y$ are positive, since neither rule generates negative numbers. So, $x < n$ and $y < n$, and so we know $x$ and $y$ are in $A$, by strong inductive hypothesis. Since $(3|n)$ and $(3|y)$, we have $(3|(x+y))$, thus $x+y \in A$. 