

# CPS102 DISCRETE MATHEMATICS

## Practice Final Exam

In contrast to the homework, no collaborations are allowed. You can use all your notes, calculator, and any books you think are useful. Write legibly and formulate each answer concisely, using only the space provided on this handout.

Your name: \_\_\_\_\_

	credit	max
Question 1		10
Question 2		10
Question 3		10
Question 4		10
Question 5		10
Question 6		10
Question 7		10
Question 8		10
Question 9		10
Question 10		10
Total		100

**Question 1.**

Prove or disprove: there exists a prime  $p > 3$  such that  $p + 2$  and  $p + 4$  are also prime.

**Question 2.**

Give tight asymptotic bounds (in terms of  $\Theta$ ) for the following recurrence using the master theorem.

$$(1) \quad T(n) = 16T(n/4) + \log^2 n$$

Or try the easier version

$$(2) \quad T(n) = 16T(n/4) + n^2$$

**Question 3.**

Function  $A(x, y, z)$  for  $x, y, z \geq 0$  is defined as follows:

1.  $A(x, y, 0) = y + 1$
2.  $A(x, 0, 1) = x$
3.  $A(x, 0, 2) = 0$
4.  $A(x, 0, z + 3) = 1$
5.  $A(x, y, z) = A(x, A(x, y - 1, z), z - 1)$

Prove that

- (a)  $A(x, y, 1) = x + y$
- (b)  $A(x, y, 2) = xy$
- (c)  $A(x, y, 3) = x^y$

**Question 4.**

We draw cards from an ordinary deck of 52 playing cards. The cards are to be drawn successively at random and without replacement. What is the probability that the second heart appears on the fifth draw?

**Question 5.**

A tournament is a simple directed graph such that if  $u$  and  $v$  are distinct vertices in the graph, exactly one of  $(u, v)$  and  $(v, u)$  is an edge of the graph. Assume all vertices are labeled.

How many different tournaments are there with  $n$  vertices?

**Question 6.**

We hash  $n$  keys into  $k = 1000$  memory locations one by one. What is the probability that the first  $i$  records do not produce a collision? Assume each key is independently and uniformly hashed into the memory locations.

**Question 7.**

A pair of dice is rolled in a remote location and an honest observer informs us that at least one of the dice came up six. What is the probability that the sum of the numbers that came up on the two dice is seven, given the information provided by the honest observer?

**Question 8.**

Give a formula for the coefficient of  $x^k$  in the expansion of  $(x - 1/x)^{100}$  where  $k$  is an integer.

**Question 9.**

Use Fermat's Little Theorem to compute  $3^{302} \pmod{5}$ .

**Question 10.**

True or false, are the following logically equivalent?  $\iff$  represents equivalence.

(a)  $\neg(p \oplus q) \iff (p \leftrightarrow q)$ .

(b)  $\neg(p \leftrightarrow q) \iff (\neg p \leftrightarrow q)$ .

(c)  $[(p \Rightarrow q) \Rightarrow r] \iff [p \Rightarrow (q \Rightarrow r)]$ .

(d)  $[\neg p \Rightarrow (q \Rightarrow r)] \iff [q \Rightarrow (p \vee r)]$ .