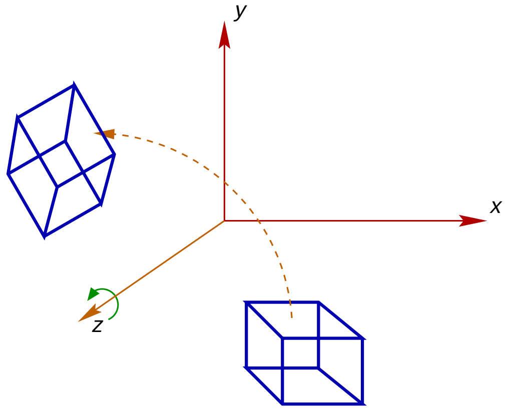
Rotation in 3D

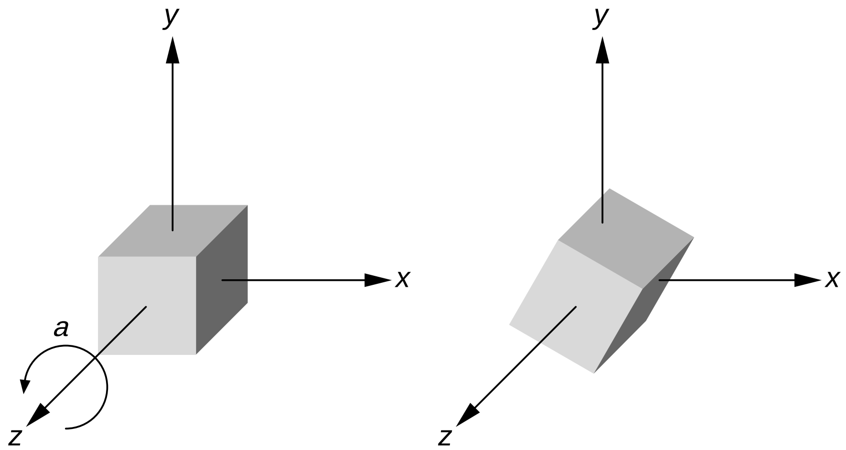
Rotation w.r.t. a line: Axis of rotation

Axis of rotation: z-axis



$$\begin{aligned} x' &= x\cos\theta - y\sin\theta \\ y' &= x\sin\theta + y\cos\theta \\ z' &= z \end{aligned}$$

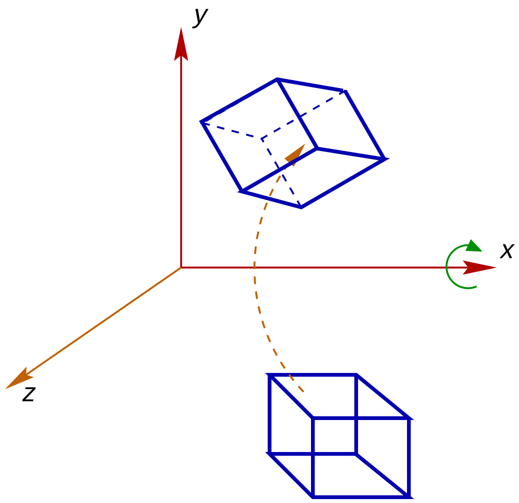
$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{R_z}(\theta)} \cdot \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$



(a) (b)

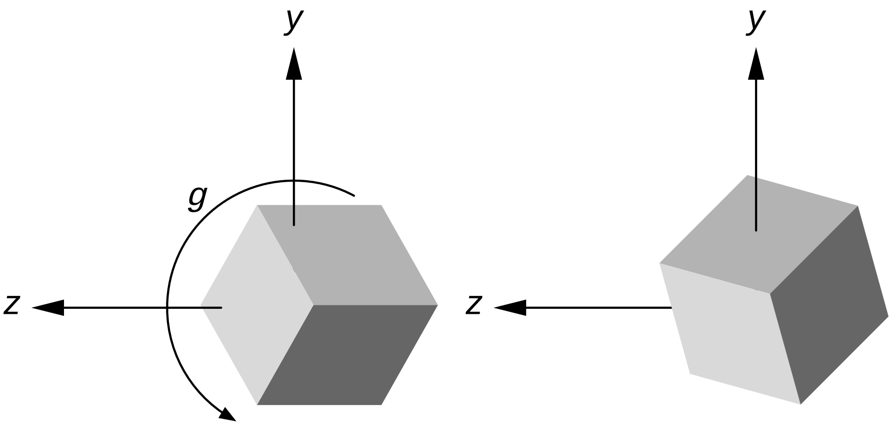
Rotation w.r.t. x-axis

Substitute $x \to y, y \to z, z \to x$.



$$\begin{aligned} y' &= y\cos\theta - z\sin\theta \\ z' &= y\sin\theta + z\cos\theta \\ x' &= x \end{aligned}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{R_x}(\theta)} \cdot \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

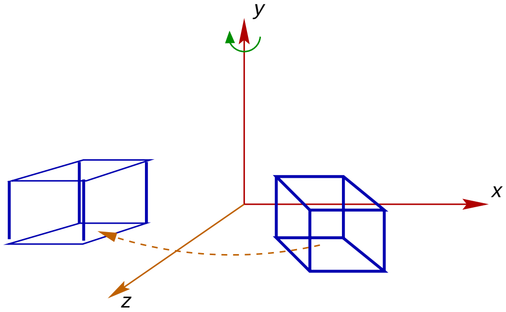


(a) (b)

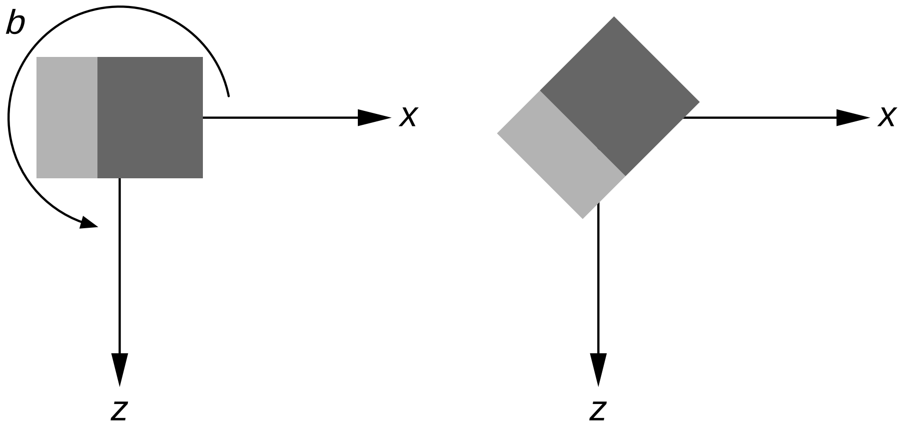
Rotation w.r.t. y-axis

Substitute $x \rightarrow y, y \rightarrow z, z \rightarrow x$ in the previous equations.

$$\begin{aligned} z' &= z\cos\theta - x\sin\theta \\ x' &= z\sin\theta + x\cos\theta \\ y' &= y \end{aligned}$$

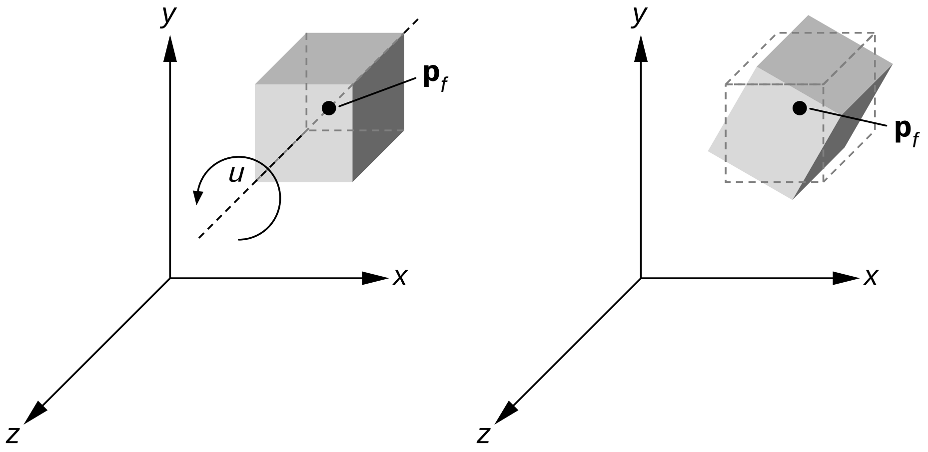


$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{R_y}(\theta)} \cdot \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

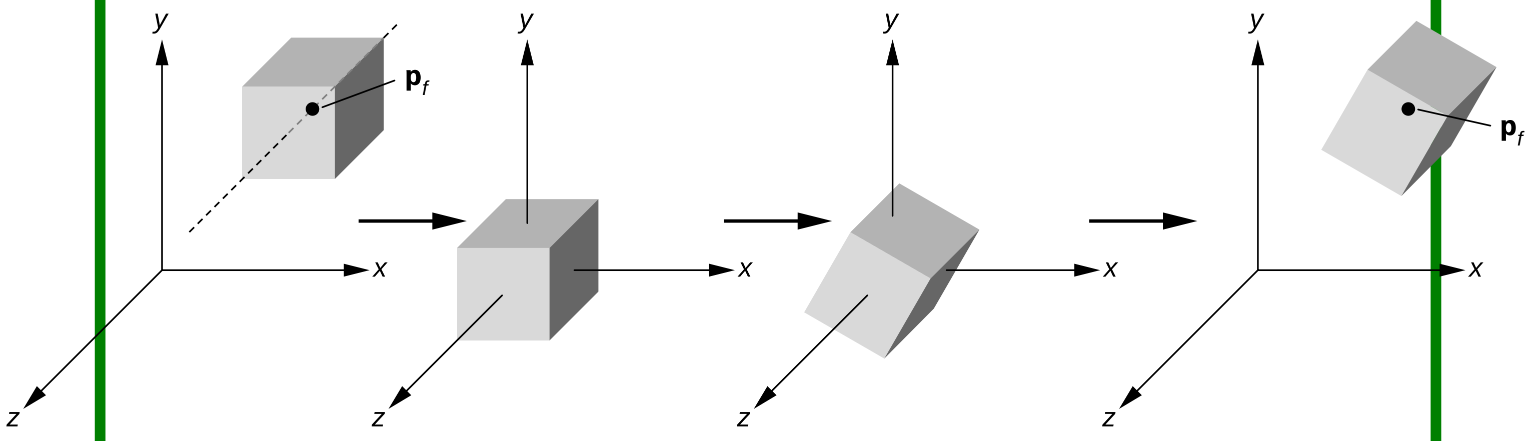


(a) (b)

Rotation wrt a Point



(a) (b)



$$R(\theta, p) = T(p) \cdot R_z(\theta) \cdot T(-p)$$

Rotation in 3D

Rotation by θ with respect to an arbitrary line $\overline{uv}$.

$$u = (u_1, u_2, u_3), \quad v = (v_1, v_2, v_3).$$

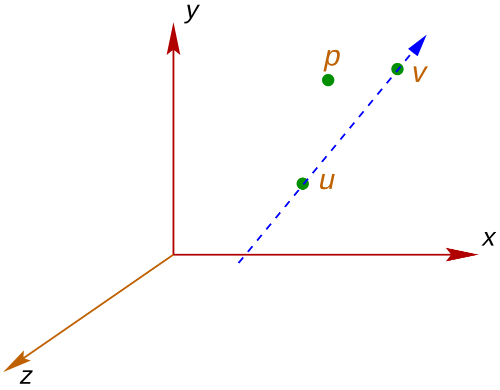
1. Translate $(-u)$ so that u becomes the origin $T(-u)$.

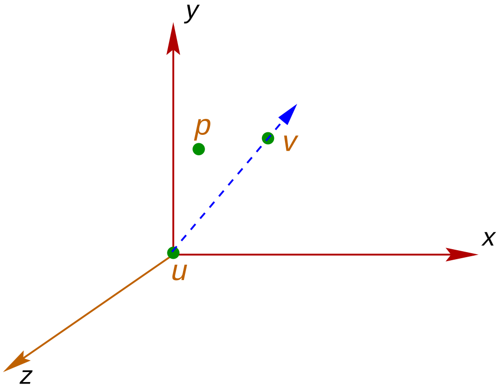
$$T(-u) = \begin{bmatrix} 1 & 0 & 0 & -u_1 \\ 0 & 1 & 0 & -u_2 \\ 0 & 0 & 1 & -u_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

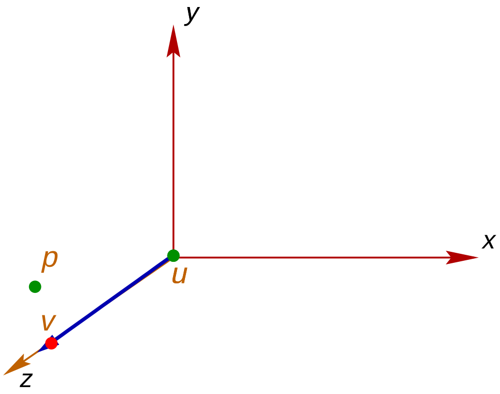
2. Perform a rotation $R_1(u, v)$ so that $\vec{uv}$ aligns with the z-axis.
3. Rotate p by angle θ w.r.t. the z-axis; $R_z(\theta)$.
4. Perform rotation $R_1^{-1}(u, v)$.
5. Translate $T^{-1}(-u) = T(u)$.

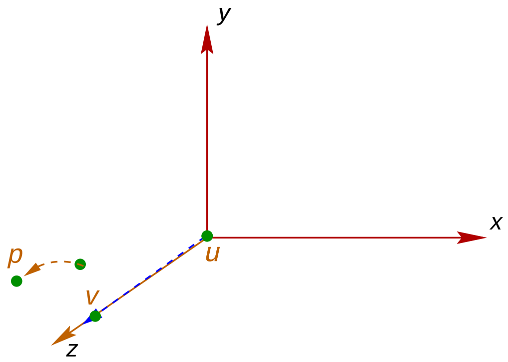
$$R(u, v, \theta) = T(u) \cdot R_1^{-1}(u, v) \cdot R_z(\theta) \cdot R_1(u, v) \cdot T(-u).$$

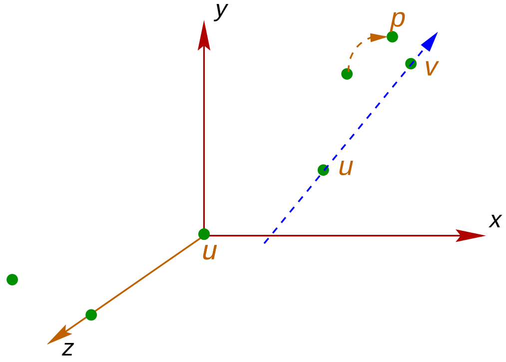
Rotation in 3D





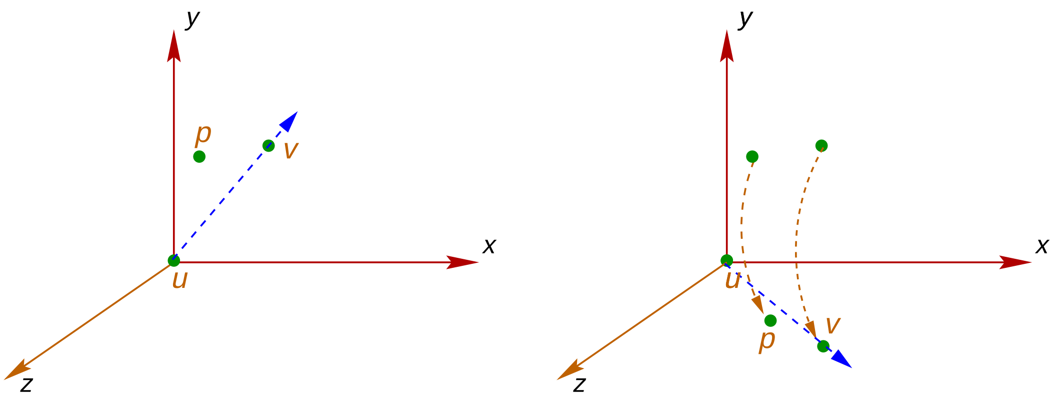




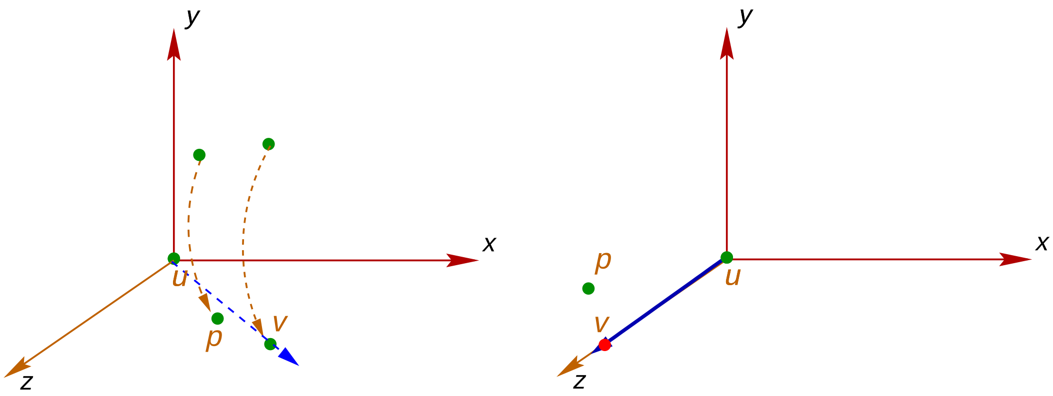


3D Rotation: Step 2

- ☆ Rotate w.r.t. x-axis so that $u\vec{v}$ lies on the xz-plane; $R_x(\alpha)$.



- ☆ Rotate w.r.t. y-axis so that $u\vec{v}$ lies on the z-axis; $R_y(\beta)$.



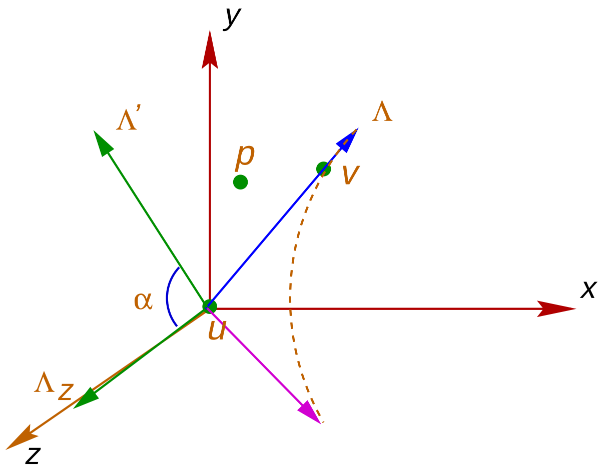
$$R(\theta) = R_x(\theta_x) \cdot R_y(\theta_y) \cdot R_x(\theta_z)$$

$\theta = (\theta_x, \theta_y, \theta_z)$: *Euler angles*

3D Rotation: Step 2

$$\begin{aligned}
\Lambda &= \frac{v-u}{|v-u|} = (a,b,c) \\
\Lambda' &= (0,b,c), yz\text{-projection of } \Lambda; \\
& |\Lambda'| = \sqrt{b^2+c^2} = d \\
\Lambda_z &= (0,0,c)
\end{aligned}$$

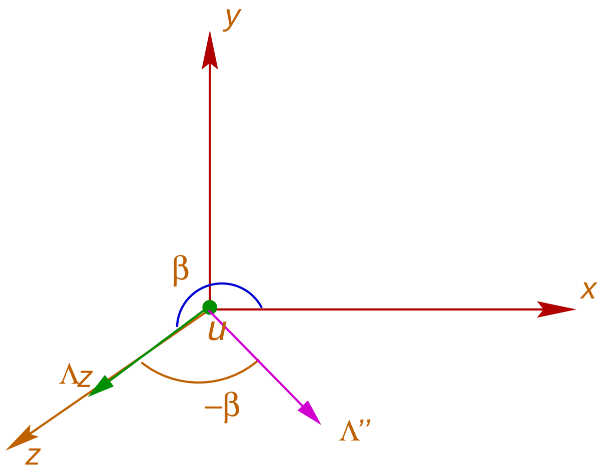
$$\begin{aligned}
\cos\alpha &= \frac{\Lambda' \cdot \Lambda_z}{|\Lambda'||\Lambda_z|} = \frac{c}{d}, \\
\sin\alpha &= \frac{b}{d}.
\end{aligned}$$



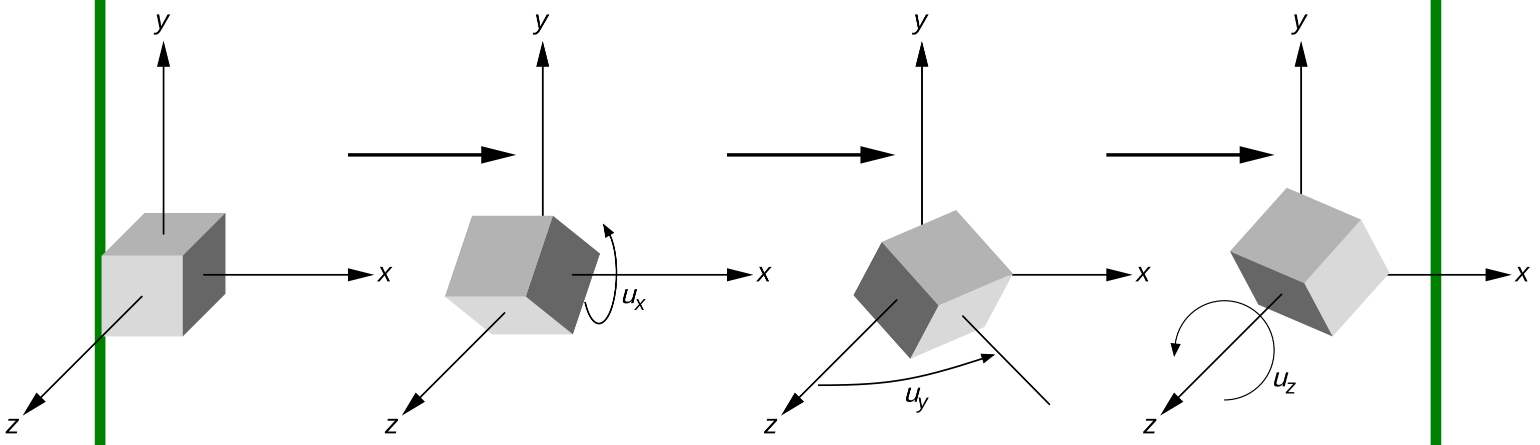
$\Lambda'' = R(\alpha)\Lambda$

β: CCW rotation around the y-axis to align Λ'' with the z-axis

$$\begin{aligned}
\cos\beta &= \frac{\Lambda'' \cdot \Lambda_z}{|\Lambda''||\Lambda_z|} = d, \\
\sin\beta &= -a.
\end{aligned}$$

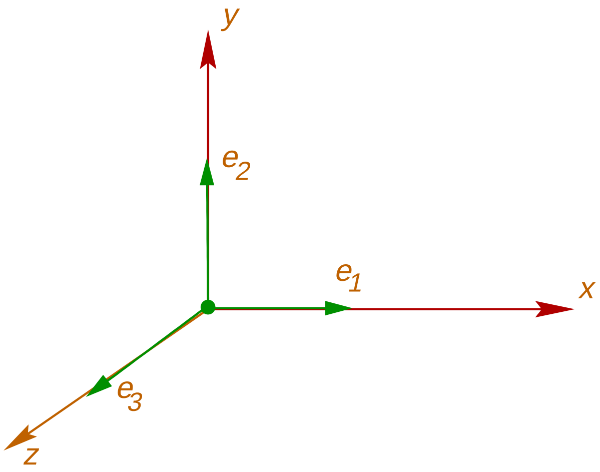


Rotation wrt a Line



$$R_x(u_x) \cdot R_y(u_y) \cdot R_z(u_z)$$

Rotation Matrices



- ☆ R: 3×3 rotation matrix.
- ☆ e_1: Unit vector along the x-axis.
- ☆ e_2: Unit vector along the y-axis.
- ☆ e_3: Unit vector along the z-axis.
- ☆ Re_i: vector after rotating e_i.
- ☆ New vectors are also unit vectors.
- ☆ The rotated vectors are normal to each other.

Rotation Matrices

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \; e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$R = [R_1 \, R_2 \, R_3]$.

R_i is the i-th column vector.

$$R \cdot e_1 = [R_1 \, R_2 \, R_3] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = R_1$$

$$R \cdot e_2 = [R_1 \, R_2 \, R_3] \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = R_2$$

$$R \cdot e_3 = [R_1 \, R_2 \, R_3] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = R_3$$

Rotation Matrices

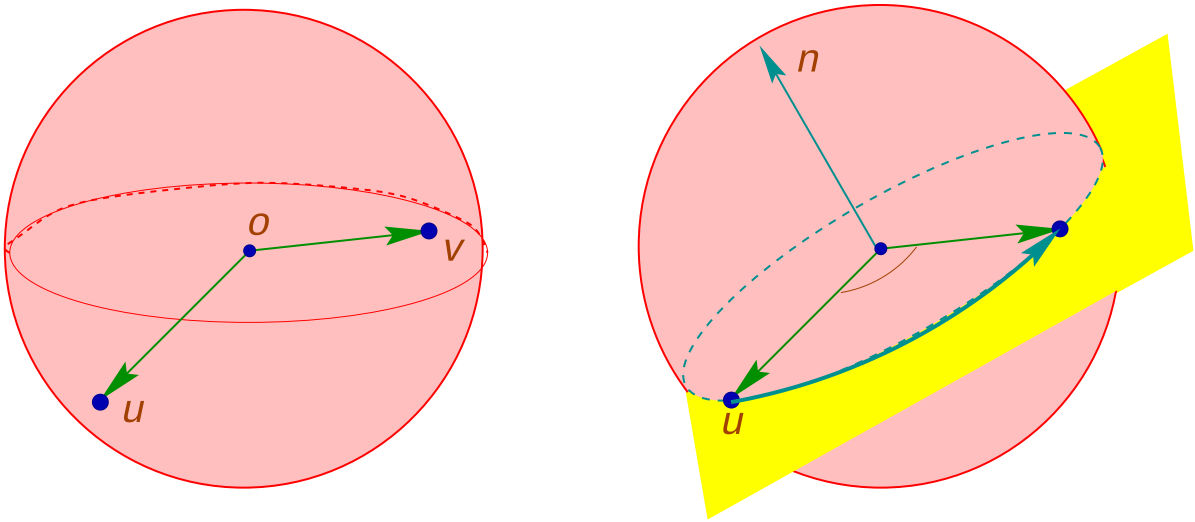
- $\|R \cdot e_i\| = \|R_i\| = 1$, for $i = 1, 2, 3$.
- $Re_1 \perp Re_2 \Rightarrow R_1 \perp R_2 \Rightarrow R_1 \cdot R_2 = 0$.
- Similarly, $R_2 \cdot R_3 = R_1 \cdot R_3 = 0$.

$$
\begin{aligned}
R^T \cdot R &= \begin{bmatrix} R_1^T \\ R_2^T \\ R_3^T \end{bmatrix} \cdot [R_1 \; R_2 \; R_3] \\
&= \begin{bmatrix} R_1 \cdot R_1 & R_1 \cdot R_2 & R_1 \cdot R_3 \\ R_2 \cdot R_1 & R_2 \cdot R_2 & R_2 \cdot R_3 \\ R_3 \cdot R_1 & R_3 \cdot R_2 & R_3 \cdot R_3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I
\end{aligned}
$$

$$R^{-1} = R^T.$$

3D Rotation

How to smoothly rotate from orientation u to v?



Performing a smooth rotating using Euler angles is difficult!

Need a method that is independent of the coordinate frame.

Quaternions

- ⋆ Discovered by William R. Hamilton in 1843.
- ⋆ Extension of complex numbers.

$$A = (\alpha, \mathbf{a})$$

- ⋆ α is a real number.
- ⋆ $\mathbf{a}$ is a three-dimensional vector.

A can also be written as

$$A = \alpha + a\mathbf{i} + b\mathbf{j} + c\mathbf{k}.$$

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

- ⋆ $\mathbf{ij} = \mathbf{k}$, $\mathbf{jk} = \mathbf{i}$, $\mathbf{ki} = \mathbf{j}$,
- ⋆ $\mathbf{ji} = -\mathbf{k}$, $\mathbf{kj} = -\mathbf{i}$, $\mathbf{ik} = -\mathbf{j}$,
- ⋆ $$\begin{aligned} A &= \alpha + a\mathbf{i} + b\mathbf{j} + c\mathbf{ij} \\ &= (\alpha + a\mathbf{i}) + (b + c\mathbf{i})\mathbf{j} \end{aligned}$$

  A is a two-dimensional complex vector.

Quaternions

- ⋆ **H**: Set of all quaternions.
- ⋆ $A = (\alpha, \mathbf{a})$, $B = (\beta, \mathbf{b})$: two quaternions.

$$\begin{aligned} A + B &= (\alpha + \beta, \mathbf{a} + \mathbf{b}) \\ A \cdot B &= (\alpha\beta - \mathbf{a} \cdot \mathbf{b}, \alpha\mathbf{b} + \beta\mathbf{a} + \mathbf{a} \times \mathbf{b}) \end{aligned}$$

- ⋆ $A \cdot (B \cdot C) = (A \cdot B) \cdot C$.
- ⋆ $\overline{A}$: conjugate of $A = (\alpha, \mathbf{a})$.
  $\overline{A} = (\alpha, -\mathbf{a})$.
- ⋆ $A \cdot \overline{A} = \overline{A} \cdot A = \alpha^2 + |\mathbf{a}|^2$.
- ⋆ $|A| = \sqrt{A \cdot \overline{A}}$.

Quaternions & 3D Rotation

- Let $Q \in \mathbf{H}$, $|Q| = 1$.
- Define $C_Q : \mathbf{H} \to \mathbf{H}$.
$$C_Q(X) = Q \cdot X \cdot \overline{Q} \quad \forall X \in \mathbf{H}.$$
- $\mathbf{n}$: Unit vector in $\mathbb{R}^3$.
- Q: $\cos(\theta/2) + \sin(\theta/2)\mathbf{n}$.
- For a vector $\mathbf{v} \in \mathbb{R}^3$, $V = (0, \mathbf{v}) \in \mathbf{H}$.
- $C_Q(V)$: $Q \cdot V \cdot \overline{Q}$

  Rotation of $\mathbf{v}$ *by angle* θ *w.r.t.* $\mathbf{n}$.
- Q, Q': two quaternions of the above form

$$\begin{aligned}
C_{Q'}(C_Q(V)) &= Q' \cdot (Q \cdot V \cdot \overline{Q}) \cdot \overline{Q'} \\
&= (Q' \cdot Q) \cdot V \cdot (\overline{Q} \cdot \overline{Q'}) \\
&= (Q' \cdot Q) \cdot V \cdot \overline{Q' \cdot Q} \\
&= C_{Q'Q}(V)
\end{aligned}$$