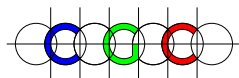


# Modeling Complex Objects

- ★ Parametric curves
- ★ Parametric surfaces
- ★ Fractal models
- ★ Grammar based models



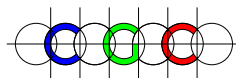
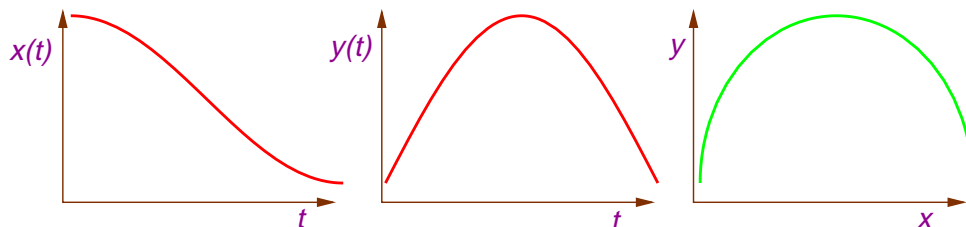
# Parametric Curves

$$Q(t) = [x(t), y(t), z(t)] \quad t \in [0, 1]$$

$$Q(t) : [0, 1] \rightarrow \mathbb{R}^3$$

## Examples:

- ★  $Q(t) : [t, t + 4, t \Leftrightarrow 5]$
- ★  $Q(t) : [t + 3, 4t + 4, 8t \Leftrightarrow 5]$
- ★  $Q(t) : [\cos(2\pi t), \sin(2\pi t), 0]$
- ★  $Q(t) : [\cos(\pi t), \sin(\pi t) + 10, 6]$



## Parametric Curves

$\deg(Q)$ :  $\max\{\deg(x(t)), \deg(y(t)), \deg(z(t))\}$

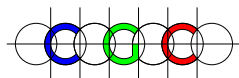
$\text{order}(Q)$ :  $\deg(Q) + 1$

A curve of order  $n$  can pass through an arbitrary set of  $n$  points.

Want to control endpoints and the tangent vectors at endpoints.

Higher degree implies more flexibility, but expensive to evaluate.

Describe a shape as a sequence of arcs.



# Continuity

## Parametric continuity

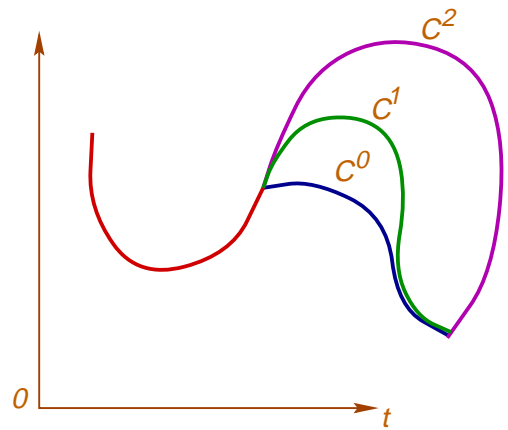
$\frac{d^n}{dt} f(t)$  continuous  $\Rightarrow f(t)$  is  $C^n$ -continuous

$C^0$ : Function is continuous

$C^\infty$ : Function is *smooth*

★  $Q_1, Q_2$ : Two curves

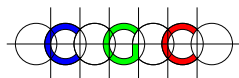
★  $Q = Q_1 \circ Q_2$   $C^n$  continuous if



$$\frac{d^n}{dt} Q_1(t) \Big|_{t=1} = \frac{d^n}{dt} Q_2(t) \Big|_{t=0}$$

## Geometric Continuity

$$\frac{d^n}{dt} Q_1(t) \Big|_{t=1} = k \frac{d^n}{dt} Q_2(t) \Big|_{t=0} \quad k > 0$$



## Parametric Cubic Curves

$$Q(t) = [x(t), y(t), z(t)]$$

$$x(t) = a_x t^3 + b_x t^2 + c_x t + d_x$$

$$y(t) = a_y t^3 + b_y t^2 + c_y t + d_y$$

$$z(t) = a_z t^3 + b_z t^2 + c_z t + d_z$$

$$T = [t^3 \ t^2 \ t \ 1]$$

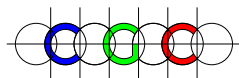
$$C = \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix}$$

$$Q(t) = T \cdot C$$

$$\frac{\partial}{\partial t} Q(t) = [x'(t) \ y'(t) \ z'(t)]$$

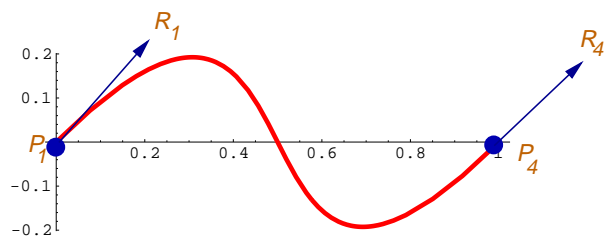
$$= [3t^2 \ 2t \ 1 \ 0] \cdot C$$

$C$  specifies the curve!



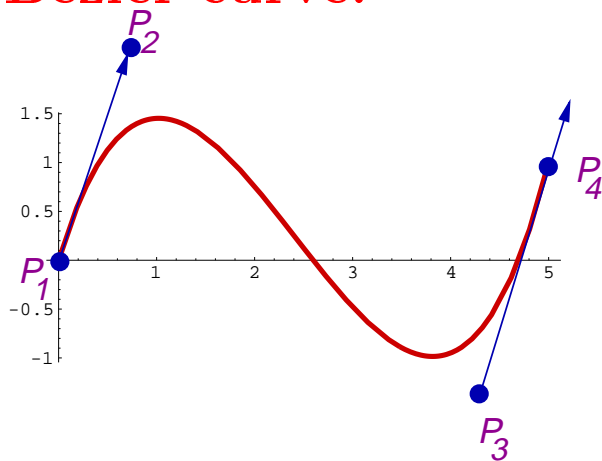
# Parametric Cubic Curves

## Hermite curve:

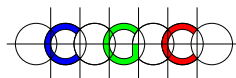


$$\begin{bmatrix} P_1 \\ P_4 \\ R_1 \\ R_4 \end{bmatrix}$$

## Bezier curve:

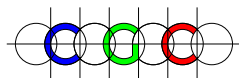


$$\begin{aligned} Q(0) &= P_1 \\ Q(1) &= P_4 \\ Q'(0) &= 3(P_2 \ominus P_1) \\ Q'(1) &= 3(P_4 \ominus P_3) \end{aligned}$$



## Basis Matrix

- ★ Definition of  $C$  has two components
  - How is a curve specified?
    - \* Four points lying on the curve.
    - \* Two endpoints and two tangent vectors.
  - Geometry of the curve (**control points**)
    - \* Coordinates of four points.
    - \* Coordinates of endpoints and values of tangent vectors.
- ★  $C = M \cdot G \quad Q(t) = T \cdot M \cdot G$
- ★  $M$ :  $4 \times 4$  matrix, called *basis matrix*  
Encodes the specification of the curve
- ★  $G$ :  $4 \times 1$  matrix, called *geometry vector*  
Specifies the geometry of four control points



## Blending Vectors

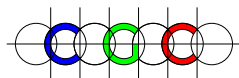
$$B = [B_1 \ B_2 \ B_3 \ B_4] = T \cdot M$$

$$Q(t) = [x(t) \ y(t) \ z(t)] = T \cdot M \cdot G = B \cdot G$$

$$\begin{aligned} &= [t^3 \ t^2 \ t \ 1] \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{bmatrix} \\ &= B_1 G_1 + B_2 G_2 + B_3 G_3 + B_4 G_4 \end{aligned}$$

- ★  $Q(t)$  is the weighted sum of  $G_i$ 's.
- ★ Weights are specified by blending functions.
- ★  $G_i^x$ :  $x$ -component of  $G_i$ .

$$\begin{aligned} x(t) &= [t^3 \ t^2 \ t \ 1] \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} G_1^x \\ G_2^x \\ G_3^x \\ G_4^x \end{bmatrix} \\ &= B_1 G_1^x + B_2 G_2^x + B_3 G_3^x + B_4 G_4^x. \end{aligned}$$



## An Example: Line Segment

$$G_1 = t(0) = (g_{11}, g_{12}), \quad G_2 = t(1) = (g_{21}, g_{22})$$



$$Q(t) = [t \ 1] \cdot \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \cdot \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}$$

$$Q(0) = G_1 = [0 \ 1] \cdot M \cdot G$$

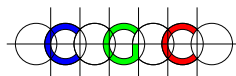
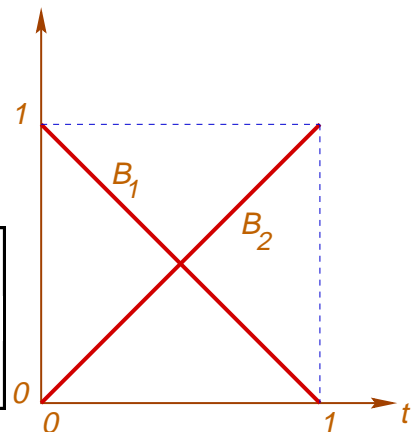
$$Q(1) = G_2 = [1 \ 1] \cdot M \cdot G$$

$$G = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} M \cdot G$$

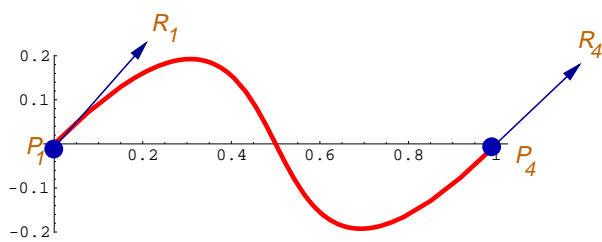
$$M = \begin{bmatrix} \Leftrightarrow 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$Q(t) = [t \ 1] \cdot \begin{bmatrix} \Leftrightarrow 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}$$

$$B_1(t) = (1 \Leftrightarrow t) \quad B_2(t) = t$$



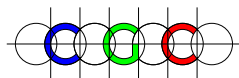
# Hermite Curve



$$\begin{bmatrix} P_1 \\ P_4 \\ R_1 \\ R_4 \end{bmatrix}$$

$$T = [t^3 \ t^2 \ t \ 1] \quad T' = [3t^2 \ 2t \ 1 \ 0]$$

$M_H$ : Basis matrix for Hermite Curve.



## Hermite Curve

$$Q(0) = P_1, Q(1) = P_4, Q'(0) = R_1, Q'(1) = R_4$$

$$Q(t) = T \cdot M_H \cdot G$$

$$Q'(t) = T' \cdot M_H \cdot G$$

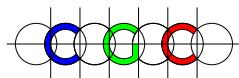
$$P_1 = [0 \ 0 \ 0 \ 1] \cdot M_H \cdot G$$

$$P_4 = [1 \ 1 \ 1 \ 1] \cdot M_H \cdot G$$

$$R_1 = [0 \ 0 \ 1 \ 0] \cdot M_H \cdot G$$

$$R_4 = [3 \ 2 \ 1 \ 0] \cdot M_H \cdot G$$

$$\begin{bmatrix} P_1 \\ P_4 \\ R_1 \\ R_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \cdot M_H G$$



# Hermite Curves

$$M_H = \begin{bmatrix} 2 & \Leftrightarrow 2 & 1 & 1 \\ \Leftrightarrow 3 & 3 & \Leftrightarrow 2 & \Leftrightarrow 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

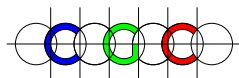
$$\begin{aligned} B(t) &= [B_1(t) \ B_2(t) \ B_3(t) \ B_4(t)] \\ &= [t^3 \ t^2 \ t \ 1] \cdot M_H \end{aligned}$$

$$B_1(t) = 2t^3 \Leftrightarrow 3t^2 + 1$$

$$B_2(t) = \Leftrightarrow 2t^3 + 3t^2$$

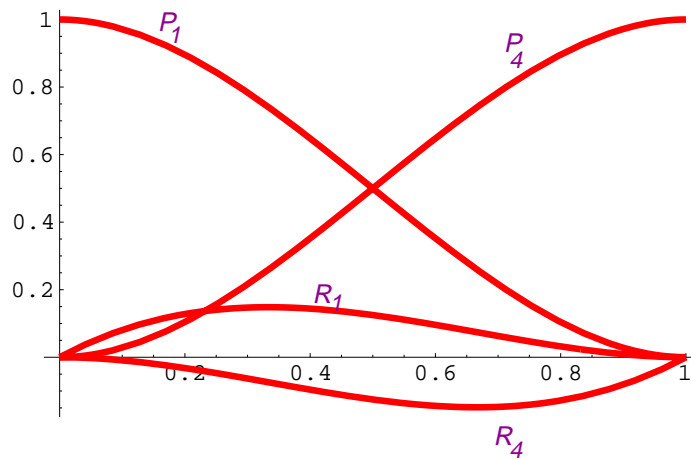
$$B_3(t) = t^3 \Leftrightarrow 2t^2 + t$$

$$B_4(t) = t^3 \Leftrightarrow t^2$$

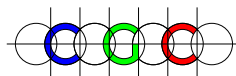
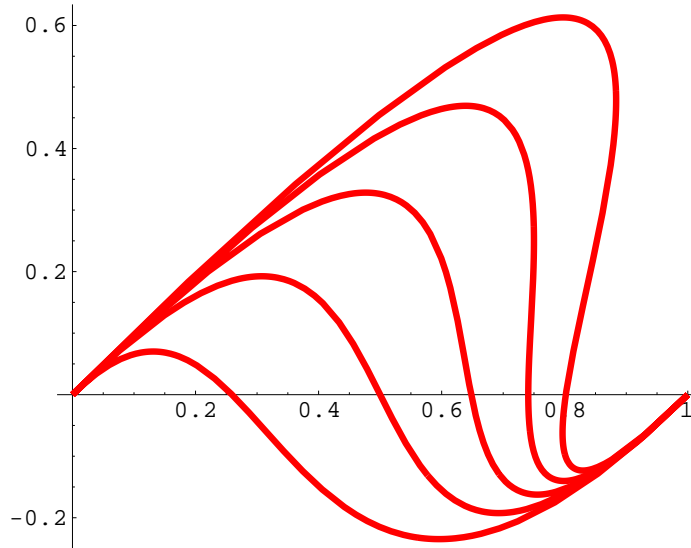


# Hermite Curves

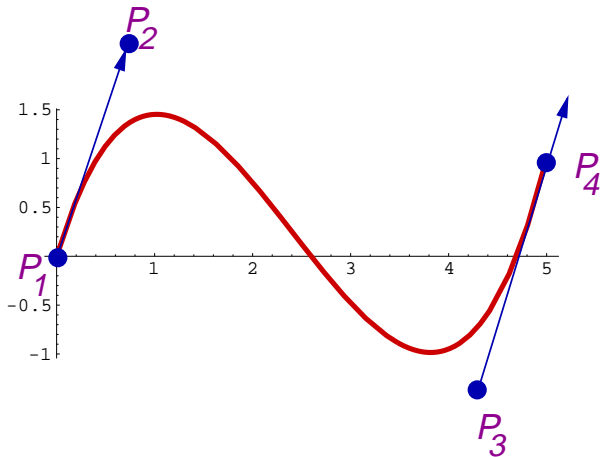
$$Q(t) = \underbrace{(2t^3 \Leftrightarrow 3t^2 + 1)}_{B_1(t)} P_1 + \underbrace{(\Leftrightarrow 2t^3 + 3t^2)}_{B_2(t)} P_4 + \underbrace{(t^3 \Leftrightarrow 2t^2 + t)}_{B_3(t)} R_1 + \underbrace{(t^3 \Leftrightarrow t^2)}_{B_4(t)} R_4$$



## Changing $R_1$



# Bezier Curves



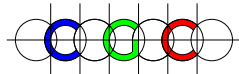
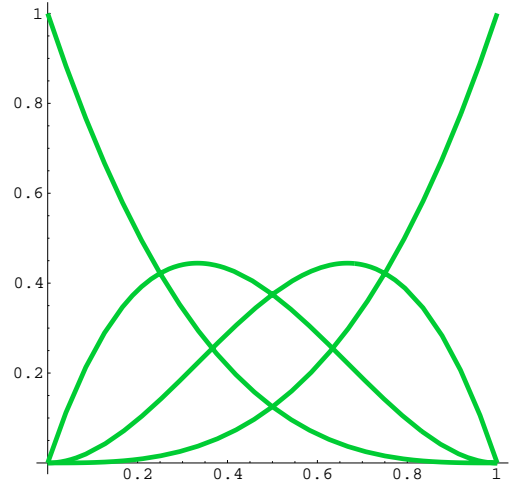
$$Q(0) = P_1$$

$$Q(1) = P_4$$

$$Q'(0) = 3(P_2 \ominus P_1)$$

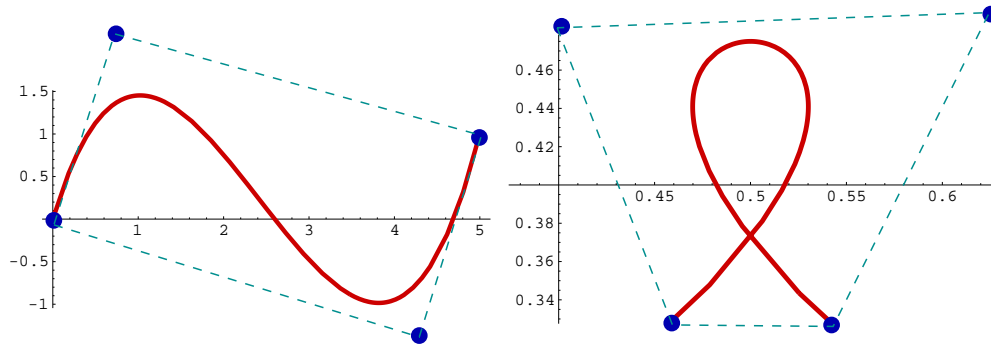
$$Q'(1) = 3(P_4 \ominus P_3)$$

$$M_B = \begin{bmatrix} \ominus 1 & 3 & \ominus 3 & 1 \\ 3 & \ominus 6 & 3 & 0 \\ \ominus 3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$



# Properties of Bézier Curves

## Convex hull property:

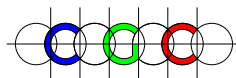
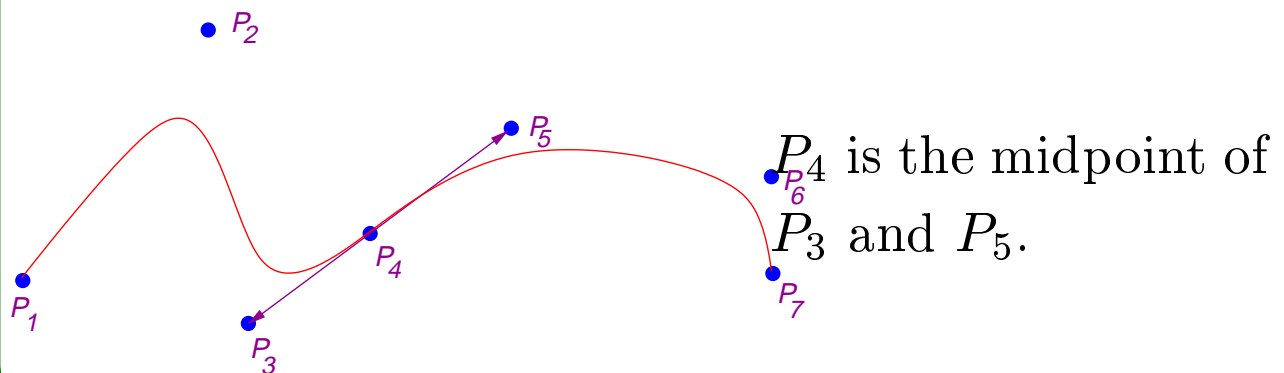


$$p \in \text{conv} (P_1, \dots, P_m) \Leftrightarrow p = \sum_{i=1}^n \lambda_i P_i$$

$$\sum_i \lambda_i = 1 \quad 0 \leq \lambda_i \leq 1$$

$$\sum_{i=1}^4 B_i(t) = 1$$

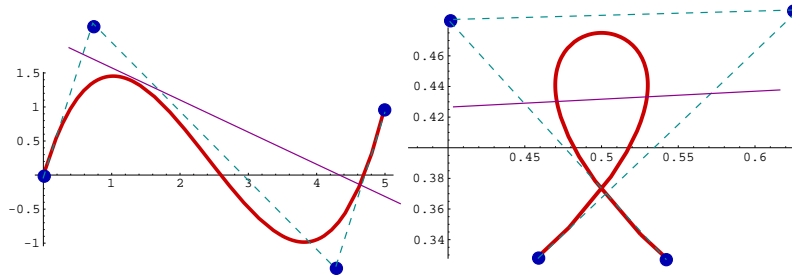
## Tangent matching:



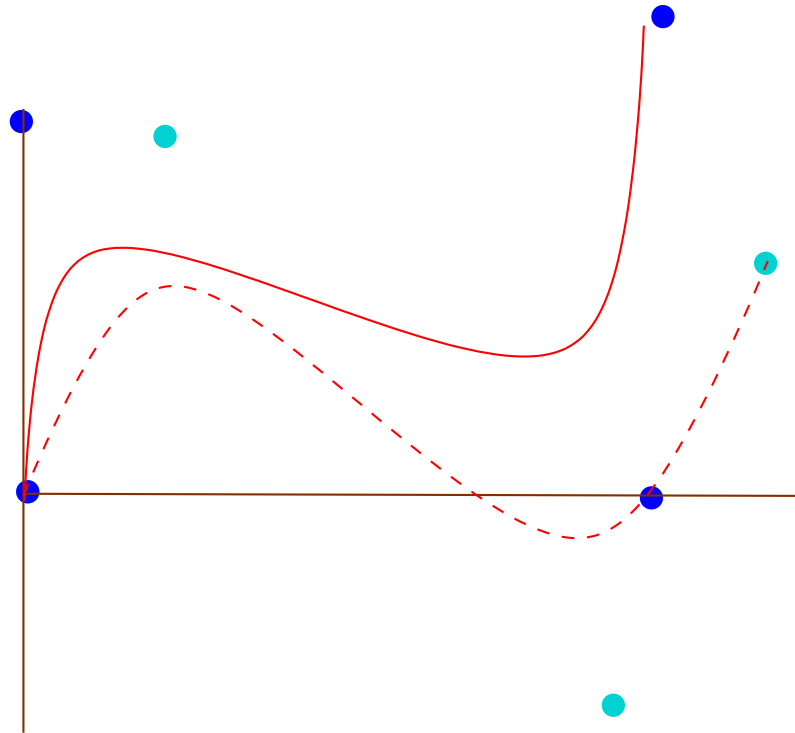
# Properties of Bézier Curves

# Crossing between a Bézier curve &  $l \leq$

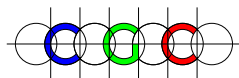
# Crossing Control polygon &  $l$



Commutates under affine transformation.



Not invariant under perspective transformation!



# Properties of Bézier Curves

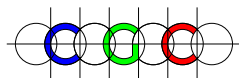
## Higher degree Bézier Curves:

Bernstein polynomial

$$B_i^n(t) = \frac{n!}{i!(n-i)!} t^i (1-t)^{n-i} \quad 0 \leq i \leq n$$

$$n = 1 \quad Q(t) = (1-t)P_1 + tP_2$$

$$n = 2 \quad Q(t) = (1-t)^2 P_1 + 2t(1-t)P_2 + t^2 P_3$$



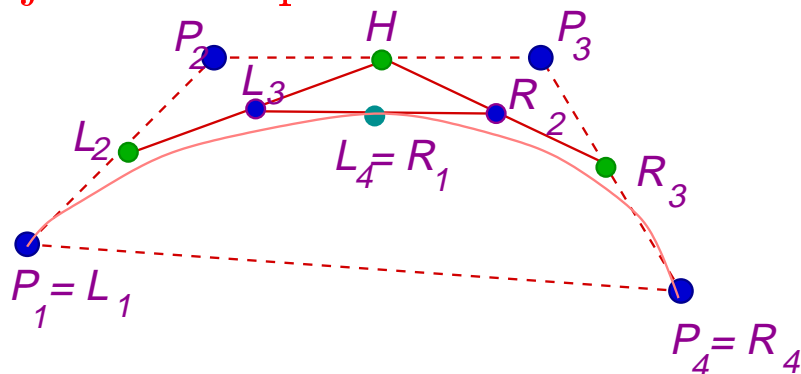
## Subdividing Bézier Curves

$$Q(t) = T \cdot M_B \cdot \underbrace{[P_1 \ P_2 \ P_3 \ P_4]^T}$$

$$Q_L(t) = T \cdot M_B \cdot \underbrace{[L_1 \ L_2 \ L_3 \ L_4]^T}$$

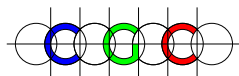
$$Q_R(t) = T \cdot M_B \cdot \underbrace{[R_1 \ R_2 \ R_3 \ R_4]^T}$$

de Casteljau technique:



$$L_2 = \frac{P_1 + P_2}{2} \quad H = \frac{P_2 + P_3}{2} \quad R_3 = \frac{P_3 + P_4}{2}$$

$$L_3 = \frac{L_2 + H}{2} \quad R_2 = \frac{H + R_3}{2} \quad L_4 = \frac{L_3 + R_2}{2}$$



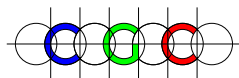
## B-Splines

### ★ Disadvantages of Hermite, Bézier Curves:

- Asymmetry
- Nonlocal effect of control points
- Cannot guarantee  $C^2$  continuity

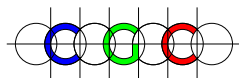
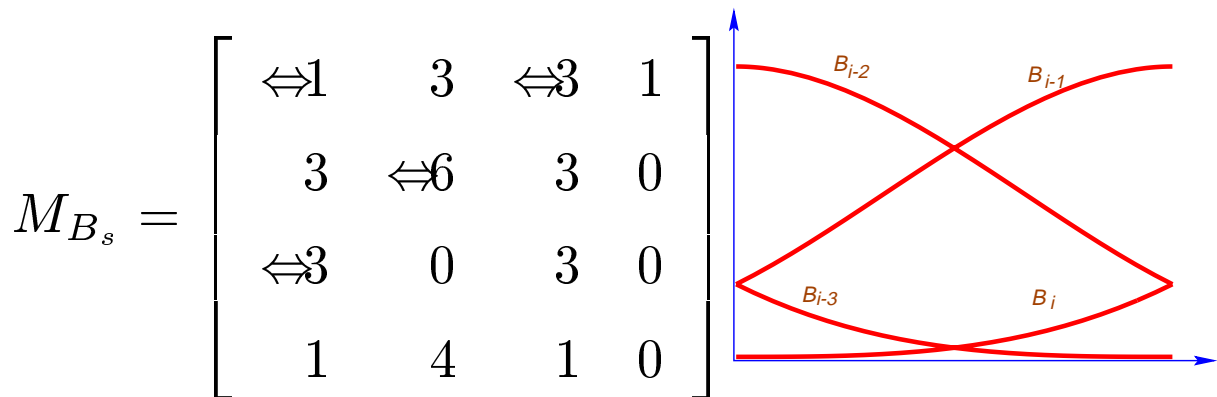
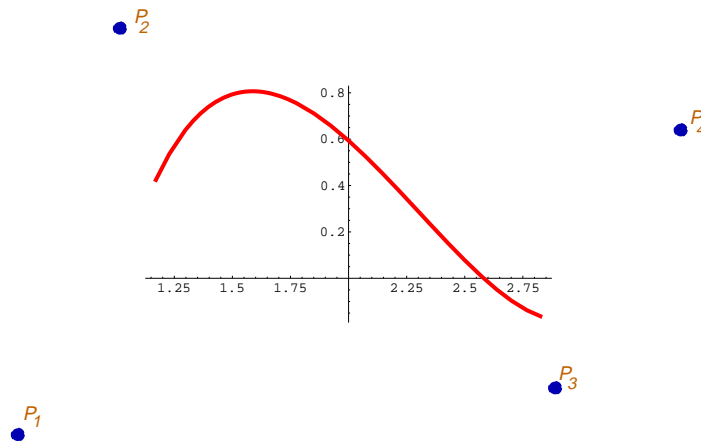
### ★ B-spline

- Simulates metal strips used in drafting.
- Provides  $C^2$  continuity.
- $P = (P_0, P_1, \dots, P_m)$ : sequence of  $m + 1$  points
- Use  $m \div 2$  B-spline curve segments  $Q_2, \dots, Q_m$  to fit a curve through  $P$ .
- Each curve is defined by 4 points:  
 $P_{i-3}, P_{i-2}, P_{i-1}, P_i$ .
- Each point controls four curve segments.

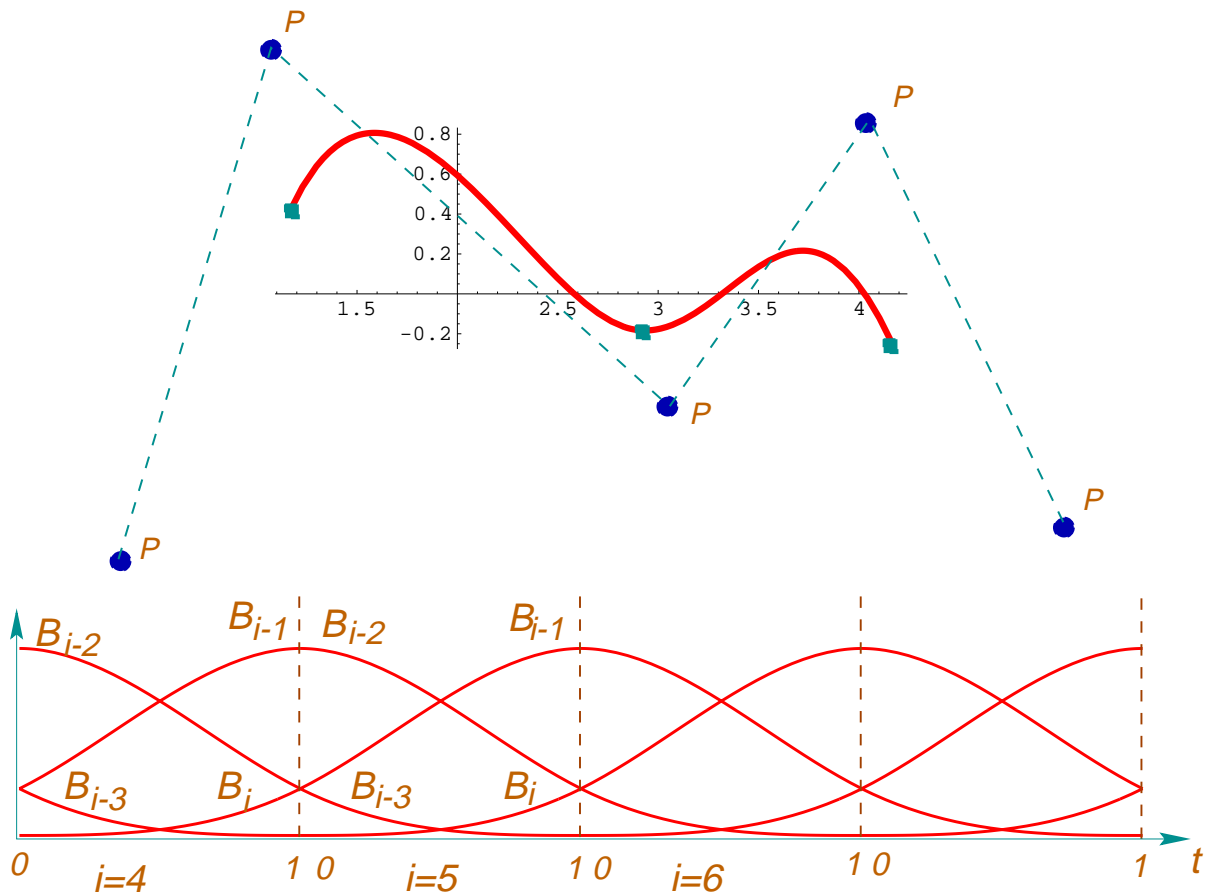


# B-Splines

$$Q_i(t) = \frac{1}{6} \left[ (1-t)^3 P_{i-3} + (3t^3 - 6t^2 + 4) P_{i-2} + (3t^3 + 3t^2 + 3t + 1) P_{i-1} + t^3 P_i \right]$$



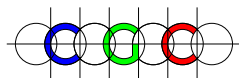
# B-Splines



**Symmetry** Each point plays the same role.

**Locality** Each point controls a local portion of the curve.

**Flexibility** Curve is  $C^2$  continuous. everywhere

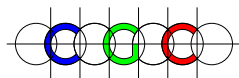


## Nonuniform, Rational Cubic Curves

- ★ Define control points using homogeneous coordinates.
- ★  $Q(t) = [X(t), Y(t), Z(t), W(t)]$   
Curve in homogeneous coordinates.

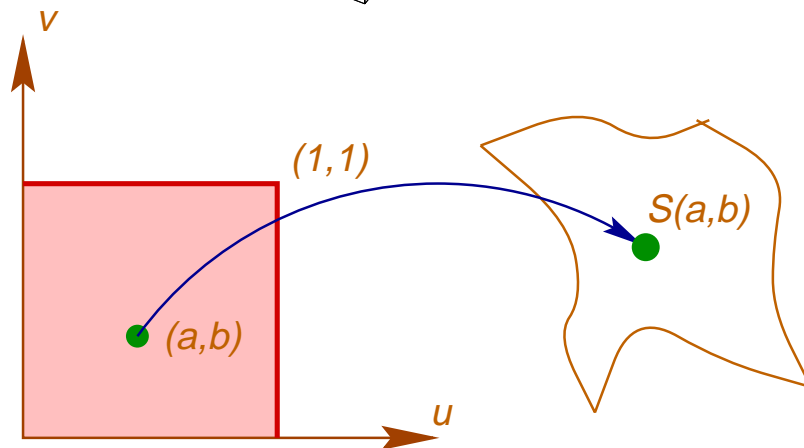
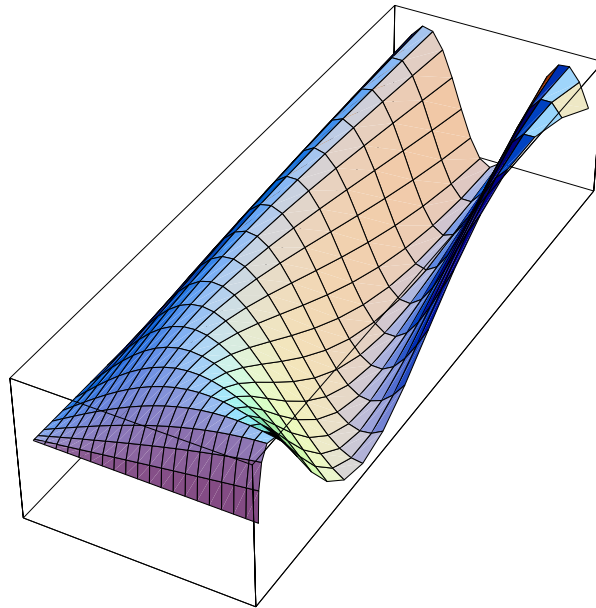
$$x(t) = \frac{X(t)}{W(t)}, y(t) = \frac{Y(t)}{W(t)}, z(t) = \frac{Z(t)}{W(t)}$$

- ★  $X, Y, Z, W$  are cubic polynomials curves.
- ★ Curves: Hermite, Bézier, B-spline.
- ★ *NURBS*: curve is B-spline.
- ★ Invariant under affine and perspective transformation.

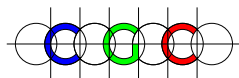


# Parametric Surfaces

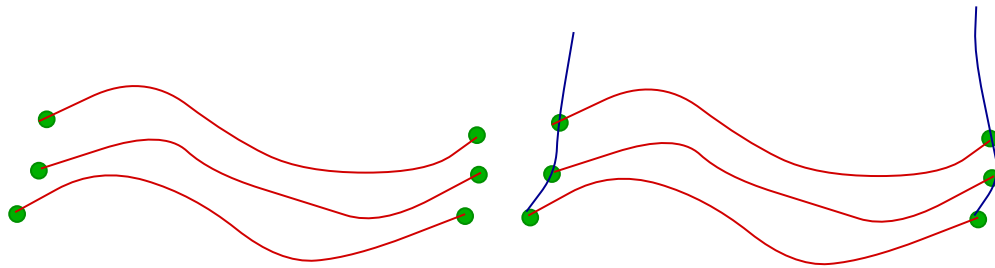
$$S(u, v) = [x(u, v), y(u, v), z(u, v)] \quad S : [0, 1]^2 \rightarrow \mathbb{R}^3$$



- ★ Interpolate corner points
- ★ Control tangent vectors
- ★ Local control



# Parametric Surfaces



- ★ Formed by connecting curves side by side.
- ★  $u$  specifies a curve  $C[u]$  in the stack.
- ★  $v$  specifies a point on  $C[u]$ .

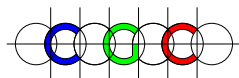
*Parametric curve:*  $Q(t) = T \cdot M \cdot G$ .

$G$  is a set of points.

*Parametric surfaces:* Regard  $G$  as a set of curves.

$$S(u, v) = U \cdot M \cdot \begin{bmatrix} G_1(v) \\ \vdots \\ G_4(v) \end{bmatrix}$$

$$G_i(v) = V \cdot M \cdot \begin{bmatrix} G_1 \\ \vdots \end{bmatrix}$$



# Bilinear Surfaces

$$L(P_1, P_2, v) = (1 - v)P_1 + vP_2$$

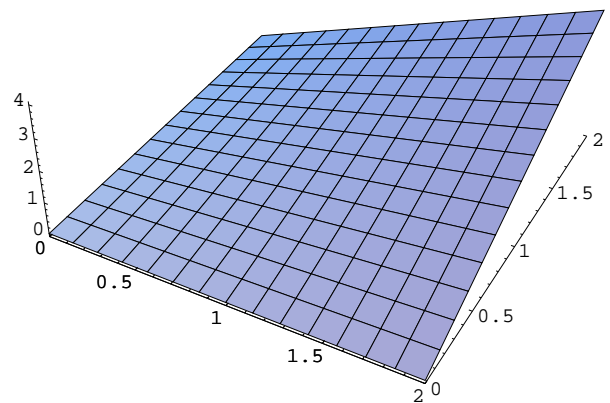
$$S(u, v) = L(L(P_1, P_2, v), L(P_3, P_4, v), u)$$

$$P_1 = (0, 0, 0)$$

$$P_2 = (1, 0, 0)$$

$$P_3 = (0, 1, 0)$$

$$P_4 = (1, 1, 1)$$

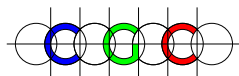
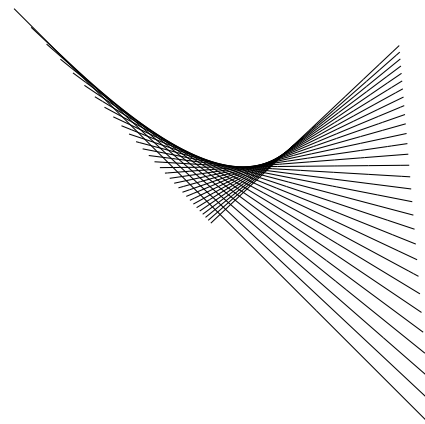


$$P_1 = (0, 0, 1)$$

$$P_2 = (1, 0, 0)$$

$$P_3 = (0, 1, 0)$$

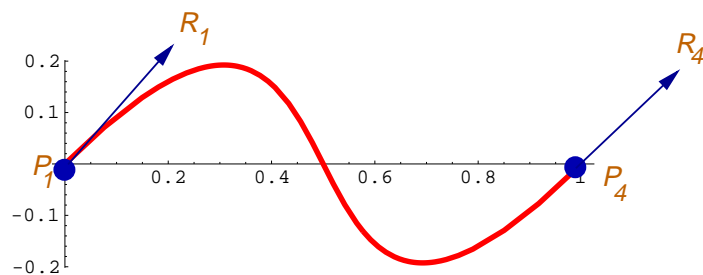
$$P_4 = (1, 1, 1)$$



# Hermite Surfaces

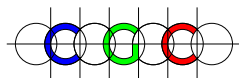
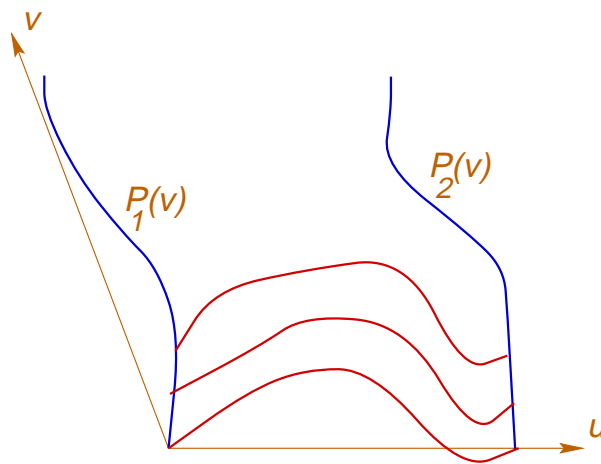
Hermite curves:

$$Q(t) = HC(G_1, G_4, G_3, G_4, t)$$



Hermite surfaces:

- ★ Stack of Hermite curves, each parametrized by  $u$ .
- ★ Control parameters of each curve in the stack is a Hermite curve, parametrized by  $v$ .



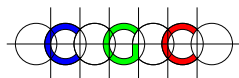
## Hermite Surfaces

$$S(u, v) = HC \left( \begin{array}{l} HC(g_{11}, g_{12}, g_{13}, g_{14}, v), \\ HC(g_{21}, g_{22}, g_{23}, g_{24}, v), \\ HC(g_{31}, g_{32}, g_{33}, g_{34}, v), \\ HC(g_{41}, g_{42}, g_{43}, g_{44}, v), \\ u \end{array} \right)$$

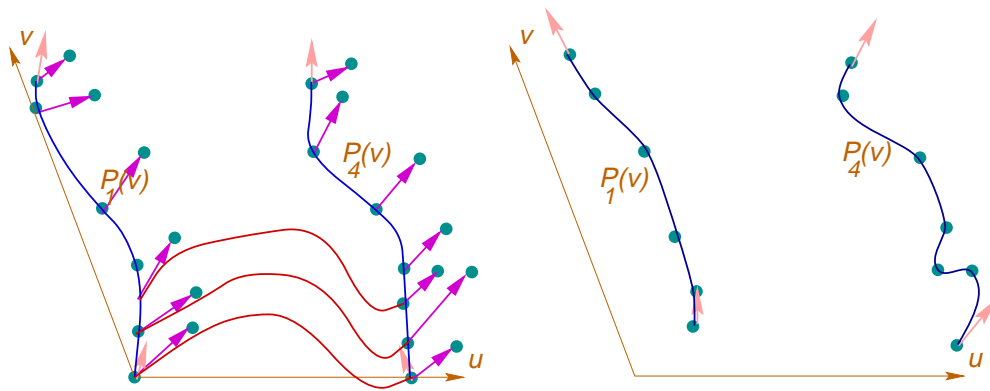
$$S(u, v) = U \cdot M_H \cdot G(v) \quad U = [u^3 \ u^2 \ u \ 1]$$

$$S(u, v) = U \cdot M_H \cdot \mathbf{G} \cdot M_H^T \cdot V^T$$

$$\mathbf{G} = \begin{bmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{bmatrix}$$



# Hermite Surfaces



$P_1(v)$ : Specifies  $S(0, v)$

$P_1(v)$  is defined by 4 parameters

$g_{11}, g_{12}, g_{13}, g_{14}$

★  $g_{11} = S(0, 0), g_{12} = S(0, 1)$

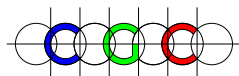
★  $g_{13} = \frac{\partial}{\partial u} S(0, v) |_{v=0}$

★  $g_{14} = \frac{\partial}{\partial u} S(0, v) |_{v=1},$

$P_4(v)$ : Specifies  $S(1, v)$

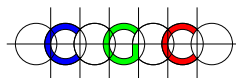
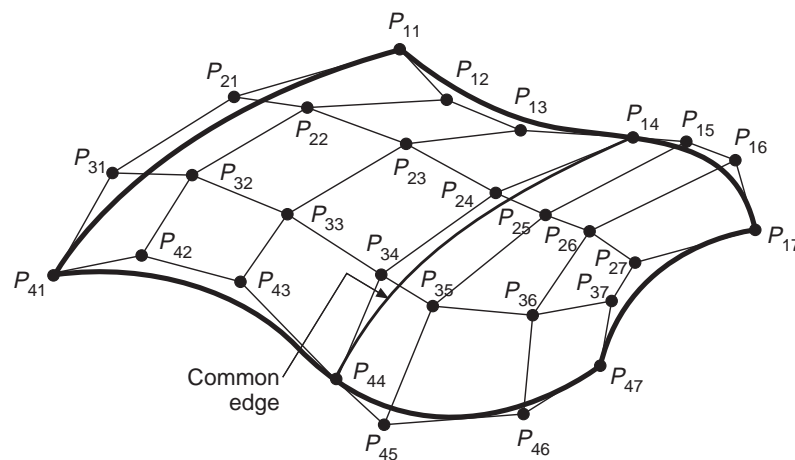
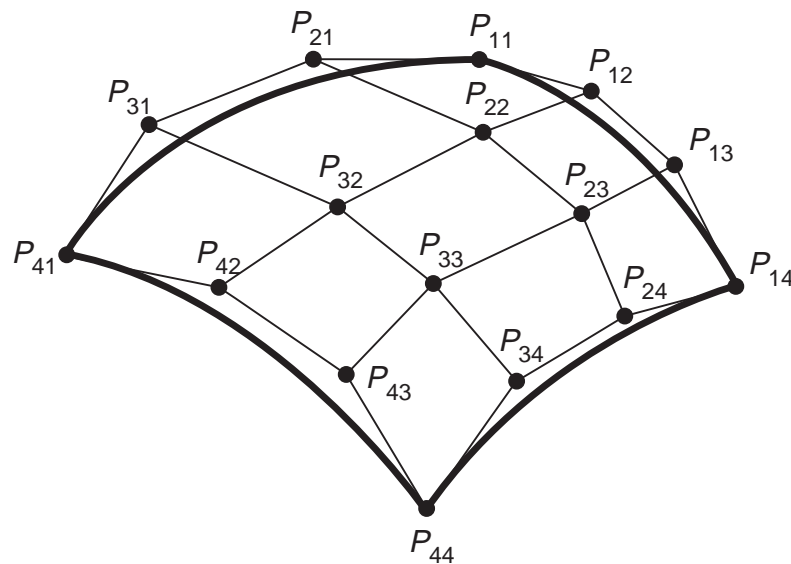
$R_1(v)$ : Specifies  $\frac{\partial}{\partial u} S(u, v) |_{u=0}$

$R_4(v)$ : Specifies  $\frac{\partial}{\partial u} S(u, v) |_{u=1}$

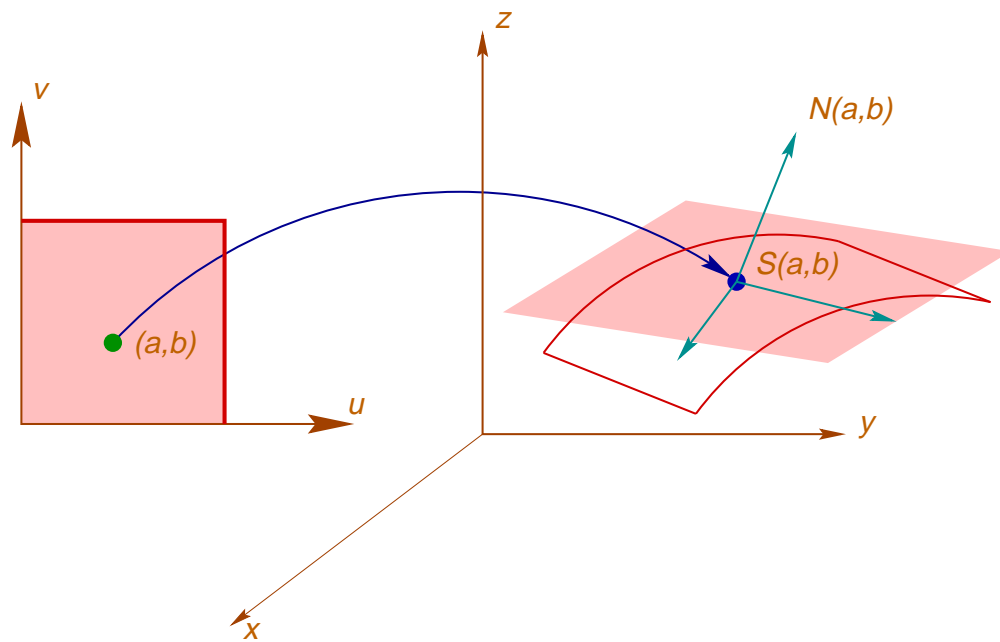


# Bezier Surfaces

$$Q(u, v) = U \cdot M_B \cdot \mathbf{G}_B \cdot M_B^T \cdot V^T$$



## Normals to Surfaces



$$\begin{aligned}\frac{\partial}{\partial u} S(u, v) &= \frac{\partial}{\partial u} U \cdot M \cdot G \cdot M^T \cdot V^T \\ &= [3u^2 \ 2u \ 1 \ 0] \cdot M \cdot G \cdot M^T \cdot V^T \\ \frac{\partial}{\partial v} S(u, v) &= U \cdot M \cdot G \cdot M^T \cdot \frac{\partial}{\partial v} V^T \\ &= U \cdot M \cdot G \cdot M^T \cdot [3v^2 \ 2v \ 1 \ 0]^T \\ N(u, v) &= \frac{\partial}{\partial u} S(u, v) \times \frac{\partial}{\partial v} S(u, v)\end{aligned}$$

