Simple Decisions

CPS 170
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Decision Theory

What does it mean to make an optimal decision?

Asked by economists to study consumer behavior
Asked by MBAs to maximize profit
Asked by leaders to allocate resources
Asked in OR to maximize efficiency of operations
Asked in AI to model intelligence
Utility Functions

A utility function is a mapping from world states to real numbers
Sometimes called a value function
Rational or optimal behavior is typically viewed as maximizing expected utility:

\[
\max_a \sum_s P(s \mid a)U(s)
\]

\(a = \text{actions}, \ s = \text{states}\)
Are Utility Functions Natural?

Some have argued that people don’t really have utility functions

What is the utility of the current state?
What was your utility at 8:00pm last night?

*Utility elicitation* is difficult problem

It’s easy to communicate *preferences*

Theorem (sorta): Given a plausible set of assumptions about your preferences, there must exist a consistent utility function
Axioms of Utility Theory

Orderability: \((A \preceq B) \land (A \succeq B) \land (A \sim B)\)

Transitivity: \((A \preceq B) \land (B \preceq C) \Rightarrow (A \preceq C)\)

Continuity: \(A \preceq B \land C \preceq \min[p, A; 1 \ominus p, C] \sim B\)

Substitutability: \(A \sim B \land [p, A; 1 \ominus p, C] \sim [p, B; 1 \ominus p, C]\)

Monotonicity: \(A \preceq B \land (p \geq q) \Rightarrow [p, A; 1 \ominus p, B] \geq [q, A; 1 \ominus q, B]\)

Decomposability: 
\([p, A; (1 \ominus p), [q, B; (1 \ominus q), C]] \sim [p, A; (1 \ominus p)q, B; (1 \ominus p)(1 \ominus q), C]\)
Consequences of Preference Axioms

Utility Principle

There exists a real-valued function $U$:

\[
U(A) > U(B) \quad \text{if} \quad A \succ B \\
U(A) = U(B) \quad \text{if} \quad A \sim B
\]

Maximum Expected Utility Principle

The utility of a lottery can be calculated as:

\[
U([p_1, S_1; K; p_n, S_n]) = \sum_i p_i U(S_i)
\]
Maximizing Utility

Suppose you want to be famous
You can be either (N,M,C)
   Nobody
   Modestly Famous
   Celebrity
Your utility function:
   \[ U(N) = 20 \]
   \[ U(M) = 50 \]
   \[ U(C) = 100 \]
You have to decide between going to grad school and becoming a professor (G) or going to Hollywood and becoming an actor (A)
Outcome Probabilities

P(N|G)=0.5, P(M|G)=0.4, P(C|G)=0.1
P(N|H)=0.6, P(M|H)=0.2, P(C|H)=0.2

Maximize expected utility:
U(N) = 20, U(M) = 50, U(C) = 100

\[ EU_G = 0.5(20) + 0.4(50) + 0.1(100) = 40 \]
\[ EU_H = 0.6(20) + 0.2(50) + 0.2(100) = 42 \]

Hollywood wins!
Utility of Money

How much happier are you with an extra $1M?
How much happier is Bill Gates with an extra $1M?

Some have proposed:

- 9
- 9.5
- 10
- 10.5
- 11
- 11.5
- 12
- 12.5
- 13
- 13.5
- 14

The graph shows the logarithmic utility of money, with the y-axis representing the utility and the x-axis representing money in dollars. The curve suggests diminishing returns as more money is accumulated.
Utility of Money

\[ U(\text{money}) \] should drop slowly in negative region too

If you’re solvent, losing $1M is pretty bad

If already $10M in debt, losing another $1M isn’t that bad

Utility of money is probably sigmoidal
A Sigmoidal Utility Function

\[ U(X) = 100 \frac{1}{1 + 2^{-0.00001X}} \]
Suppose $U(X) = X$, would you spend $1 for a 1 in a million chance of winning $1M? 

Suppose you start with c dollars: 

$$EU(\text{gamble}) = \frac{1}{1,000,000}(1,000,000 - 1 + c) + \left(1 - \frac{1}{1,000,000}\right)(c - 1) = c$$

$$EU(\text{do\_nothing}) = c$$

Starting amount doesn’t matter 
You have no expected benefit from gambling
Suppose: \[ U(X) = 100 \frac{1}{1 + 2^{-0.00001X}} \]

Suppose you start with $1M
EU(gamble)-EU(do\_nothing)=-5.7\cdot10^{-7}
Winning is worthless

Suppose you start with -$1M
EU(gamble)-EU(do\_nothing)=+4.9\cdot10^{-5}
Gambling is rational because losing doesn’t hurt
Multiattribute Utility Functions

So far, we have defined utility over *states*
As always, there are too many states

We’d like to define utility functions over variables in some clever way, just as we defined probabilities distributions over variables using Bayes nets
What’s a natural way to decompose utility?
Additive Independence

Suppose it makes me happy to have my car clean.
Suppose it makes me happy to have coffee.
\[ U = U(\text{coffee}) + U(\text{clean}) \]
It seems that these don’t interact.
However, suppose there’s a tea variable.
\[ U = U(\text{coffee}) + U(\text{tea}) + U(\text{clean}) \]
Probably not. I’d need \( U(\text{coffee,tea}) + U(\text{clean}) \)
Influence diagrams generalize Bayes nets with decisions and utilities.

Can be solved using a variable elimination type algorithm.
Value of Information

Expected utility of action $a$ with evidence $E$:

$$EU_E(A | E) = \max_a \prod_i P(S_i | E, a)U(S_i)$$

Expected utility given new evidence $E'$

$$EU_{E,E'}(A | E, E') = \max_a \prod_i P(S_i | E, E', a)U(S_i)$$

Value of knowing $E'$ (value of perfect information)

$$VPI_{E}(E') = \prod_{E'} P(E' | E)EU_{E,E'}(a | E, E')EU(a_E | E)$$

- Expected utility given
- New information
- Previous Expected utility (weighted)
Properties of VPI

VPI is non-negative!
VPI is not additive
VPI is order independent

VPI is easy to compute and is often used to determine how much you should pay for one extra piece of information. Why is this myopic?

For example, knowing X AND Y together may useful, while knowing just one alone may be useless.
More Properties of VOI

Acquiring information optimally is very difficult

Need to construct a conditional plan for every possible outcome before you ask for even the first piece of information

Suppose you’re a doctor planning to treat a patient
Picking the optimal test to do first requires that you consider all subsequent tests and all possible treatments as a result of these tests

General versions of this problem are intractable!
Conclusions

Decision theory provides a framework for optimal decision making

Principle: Maximize Expected Utility

Easy to describe in principle

Application to complex problems can require advanced planning and probabilistic reasoning techniques