Decision Theory

CPS 170
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Decision Theory
What does it mean to make an optimal decision?
• Asked by economists to study consumer behavior
• Asked by MBAs to maximize profit
• Asked by leaders to allocate resources
• Asked in OR to maximize efficiency of operations
• Asked in AI to model intelligence
• Asked (sort of) by any intelligent person every day

Utility Functions
• A utility function is a mapping from world states to real numbers
• Sometimes called a value function
• Rational or optimal behavior is typically viewed as maximizing expected utility:
  \[ \max_a \sum_s P(s \mid a) U(s) \]
  \(a = \text{actions}, \ s = \text{states}\)

Are Utility Functions Natural?
• Some have argued that people don’t really have utility functions
  • What is the utility of the current state?
  • What was your utility at 8:00pm last night?
  • Utility elicitation is difficult problem
• It’s easy to communicate preferences
• Theorem (sorta): Given a plausible set of assumptions about your preferences, there must exist a consistent utility function

Axioms of Utility Theory
• Orderability: \((A \succ B) \lor (A \prec B) \lor (A = B)\)
• Transitivity: \((A \succ B) \land (B \succ C) \Rightarrow (A \succ C)\)
• Continuity*: \(A \succ B \succ C \Rightarrow \exists \{p, A \prec p, C\} \sim B\)
• Substitutability: \(A \succ B \Rightarrow [p, A \prec p, C] \sim [p, B \prec p, C]\)
• Monotonicity*: \(A \succ B \Rightarrow [p \geq q \Rightarrow [p, A \prec p, B] \geq [q, A \prec q, B]]\)
• Decomposability:
  \[ [p, A;(1-p)q,B;(1-q),C]] \sim [p, A;(1-p)q,B;(1-p)(1-q),C]\]

Consequences of Preference Axioms
• Utility Principle
  • There exists a real-valued function U:
    \[ U(A) > U(B) \iff A \succ B \]
    \[ U(A) = U(B) \iff A \equiv B \]
• Expected Utility Principle
  • The utility of a lottery can be calculated as:
    \[ U([1_p, S_1; \ldots ; 1_p, S_n]) = \sum p U(S_i) \]
More Consequences

• Scale invariance

• Shift invariance

Maximizing Utility

• Suppose you want to be famous
• You can be either (N,M,C)
  • Nobody
  • Modestly Famous
  • Celebrity
• Your utility function:
  • U(N) = 20
  • U(M) = 50
  • U(C) = 100
• You have to decide between going to grad school and becoming a professor (G) or going to Hollywood and becoming an actor (A)

Outcome Probabilities

• P(N|G) = 0.5, P(M|G) = 0.4, P(C|G) = 0.1
• P(N|H) = 0.6, P(M|H) = 0.2, P(C|H) = 0.2
• Maximize expected utility:
  • U(N) = 20, U(M) = 50, U(C) = 100
  \[ EU_G = 0.5(20) + 0.4(50) + 0.1(100) = 40 \]
  \[ EU_H = 0.6(20) + 0.2(50) + 0.2(100) = 42 \]
• Hollywood wins!

Utility of Money

• How much happier are you with an extra $1M?
• How much happier is Bill Gates with an extra $1M?
• Some have proposed:
  \[ U(money) \text{ should drop slowly in negative region too} \]
  
  • If you’re solvent, losing $1M is pretty bad
  • If already $10M in debt, losing another $1M isn’t that bad
  • Utility of money is probably sigmoidal

A Sigmoidal Utility Function

\[ U(X) = 100 \frac{1}{1 + 2^{-0.00001X}} \]
Utility & Gambling

• Suppose $U(X) = X$, would you spend $1 for a 1 in a million chance of winning $1M? 

• Suppose you start with c dollars:
  • $EU_{\text{gambler}} = \frac{1}{1000000}(1000000-1+c) + (1-1/1000000)(c-1)=c$
  • $EU_{\text{do nothing}} = c$

• Starting amount doesn’t matter
• You have no expected benefit from gambling

Sigmoidal Utility & Gambling

• Suppose: $U(X) = \frac{1}{1 + 2^{-0.00001X}}$

• Suppose you start with $1M
  • $EU_{\text{gambler}} - EU_{\text{do nothing}} = -5.7 \times 10^{-7}$
  • Winning is worthless

• Suppose you start with -$1M
  • $EU_{\text{gambler}} - EU_{\text{do nothing}} = +4.9 \times 10^{-6}$
  • Gambling is rational because losing doesn’t hurt

Additive Independence

• Suppose it makes me happy to have my car clean
• Suppose it makes me happy to have coffee
• $U = U(\text{coffee}) + U(\text{clean})$
• It seems that these don’t interact
• However, suppose there’s a tea variable
• $U = U(\text{coffee}) + U(\text{tea}) + U(\text{clean})$??
• Probably not. I’d need $U(\text{coffee, tea}) + U(\text{clean})$

• Often implicit!

Value of Information

• Expected utility of action a with evidence E:
  $$EU_a(A_1|E) = \max_a \sum_s P(S, | E, a)U(S)$$

• Expected utility given new evidence $E’$
  $$EU_{E,E'}(A_1|E, E') = \max_a \sum_s P(S, | E, E', a)U(S)$$

• Value of knowing $E’$ (value of perfect information)
  $$VPI(E) = \left( \sum P(E'|E)EU_{E,E'}(A_1|E, E') \right) - EU(a_1|E)$$

Properties of VPI

• VPI is non-negative!
• VPI is not additive
• VPI is order independent

• VPI is easy to compute and is often used to determine how much you should pay for one extra piece of information. Why is this myopic?

For example, knowing X AND Y together may useful, while knowing just one alone may be useless.

More Properties of VOI

• Acquiring information optimally is very difficult

• Need to construct a conditional plan for every possible outcome before you ask for even the first piece of information
  • Suppose you’re a doctor planning to treat a patient
  • Picking the optimal test to do first requires that you consider all subsequent tests and all possible treatments as a result of these tests

• General versions of this problem are intractable!
Decision Theory as Search

- Can view DT probs as search probs
- States = atomic events

Max nodes

Chance

nodes

P = 0.5
P = 0.6
P = 0.4
P = 0.9
P = 0.1

DT as Search

- Attach costs to arcs, leaves
- Path(s) w/ lowest expected cost = optimal
- Minimizing expect cost = maximizing expected utility
- Expectimax

\[ V(n_{max}) = \max_{n\in\text{successors}(n)} V(s) \]
\[ V(n_{\text{chance}}) = \sum_{n\in\text{successors}(n)} V(s)p(s) \]

Optimal Solutions and Computation Cost

- Optimal solutions to decision theoretic problems are necessarily paths
- Why?
- What does this say about cost of decision theoretic reasoning?

Conclusions

- Decision theory provides a framework for optimal decision making
- Principle: Maximize Expected Utility
- Easy to describe in principle
- Application to complex problems can require advanced planning and probabilistic reasoning techniques