Due Date: October 2, 2012

Remark: Prove the correctness of every algorithm and analyze its running time.

Problem 1: Let $G = (V, E)$ be a weighted, directed graph, with $|V| = n$ and $|E| = m$, such that the weight of each edge is positive. Let $s$ and $t$ be two vertices of $G$. Note that there can be many shortest paths from $s$ to $t$ in $G$.

(i) Describe an $O((m + n) \log n)$ time algorithm to count the number of different shortest paths from $s$ to $t$ in $G$.

(ii) How fast can you do the same if edges weight are allowed to be negative? You can assume that there are no non-positive cycles.

Problem 2: For $i = 1, \ldots, k$, let $G_i = (V, E_i)$ be a sequence of $k$ graphs, i.e., all graphs have a common set of vertices, but the edge sets are different. One can think of constructing $G_i$ from $G_{i-1}$ by deleting some of its edges and inserting some new edges to it. Fix two vertices $s$ and $t$.

For $i \leq k$, let $P_i$ be a path from $s$ to $t$ in $G_i$. Let $\ell(P_i)$ be the number of edges in $P_i$, and define $\chi(P_1, \ldots, P_k)$ to be the number of indices $i$, for $1 \leq i \leq k$, for which $P_i \neq P_{i+1}$. We define the cost of $P_1, \ldots, P_k$ to be

$$\mu(P_1, \ldots, P_k) = \sum_{i=1}^{k} \ell(P_i) + \chi(P_1, \ldots, P_k).$$

(i) Suppose it is possible to choose a single path that is an $st$-path in each $G_i$. Describe a polynomial time algorithm to find the shortest such path.

(ii) Give a polynomial-time algorithm to find a sequence of paths $P_1, \ldots, P_k$ of minimum cost, where $P_i$ is a $st$-path in $G_i$.

Problem 3: For any flow network $G$ and any vertices $u$ and $v$, let bottleneck$_G(u, v)$ denote the maximum, over all paths $\pi$ in $G$ from $u$ to $v$, of the minimum-capacity edge along $\pi$. Describe an algorithm to construct a spanning tree $T$ of $G$ such that bottleneck$_T(u, v) = \text{bottleneck}_G(u, v)$. (Edges in $T$ inherit their capacities form $G$.)

Problem 4: (i) Formulate the max $st$-flow as an LP in a matrix form. Write its dual and argue that it is the same as computing a min $st$-cut.

(ii) Regard the LP formulation of max-flow as the dual LP. Given a dual feasible solution $f$, i.e., an $st$-flow, write the corresponding restricted dual LP (RDP). Show that RDP implies that if $f$ is not an optimal solution, then there is a path from $s$ to $t$ in the residual graph.
Problem 5: Suppose you are running a web site that is visited by the same set of people every day. Each visitor claims membership in one or more demographic groups; for example, a visitor might describe herself as female, 30–40 years old, a mother, an academic, etc. Your site is supported by advertisers. Each advertiser has told you which demographic groups should see its ads and how many of its ads you must show each day. Altogether, there are $n$ visitors, $k$ demographic groups, and $m$ advertisers.

Describe an efficient algorithm to determine, given all the data described in the previous paragraph, whether you can show each visitor exactly one ad per day, so that every advertiser has its desired number of ads displayed, and every ad is seen by someone in an appropriate demographic group.