Decision Theory

What does it mean to make an optimal decision?

- Asked by economists to study consumer behavior
- Asked by MBAs to maximize profit
- Asked by leaders to allocate resources
- Asked in OR to maximize efficiency of operations
- Asked in AI to model intelligence
- Asked (sort of) by any intelligent person every day

Utility Functions

- A utility function is a mapping from world states to real numbers
- Sometimes called a value function
- Rational or optimal behavior is typically viewed as maximizing expected utility:

\[
\max_a \sum_s P(s \mid a) U(s)
\]

\(a = \text{actions}, s = \text{states}\)

Are Utility Functions Natural?

- Some have argued that people don’t really have utility functions
  - What is the utility of the current state?
  - What was your utility at 8:00pm last night?
  - Utility elicitation is difficult problem
- It’s easy to communicate preferences
- Theorem (sorta): Given a plausible set of assumptions about your preferences, there must exist a consistent utility function

Axioms of Utility Theory

- Orderability: \((A \succ B) \lor (A < B) \lor (A = B)\)
- Transitivity: \((A \succ B) \land (B \succ C) \Rightarrow (A \succ C)\)
- Continuity*: \(A \succ B \succ C \Rightarrow \exists p, A \succ p \succ C \succ B\)
- Substitutability: \(A \sim B \Rightarrow [p, A \sim p, B] \sim [p, B \sim p, A]\)
- Monotonicity*: \(A \succ B \Rightarrow [p \geq q \Rightarrow [p, A \sim p, B] \geq [q, A \sim q, B]]\)
- Decomposability: \([p, A; (1-p), [q, B; (1-q), C]] \sim [p, A; (1-p)q, B; (1-p)(1-q), C]\)

Consequences of Preference Axioms

- Utility Principle
  - There exists a real-valued function \(U:\)
    \[U(A) > U(B) \Leftrightarrow A \succ B\]
    \[U(A) = U(B) \Leftrightarrow A = B\]
- Expected Utility Principle
  - The utility of a lottery can be calculated as:
    \[U([p_1, S_1; \ldots; p_n, S_n]) = \sum p_i U(S_i)\]
More Consequences

- Scale invariance
- Shift invariance

Maximizing Utility

- Suppose you want to be famous
- You can be either (N,M,C)
  - Nobody
  - Modestly Famous
  - Celebrity
- Your utility function:
  - \( U(N) = 20 \)
  - \( U(M) = 50 \)
  - \( U(C) = 100 \)
- You have to decide between going to grad school and becoming a professor (G) or going to Hollywood and becoming an actor (A)

Outcome Probabilities

- \( P(N|G) = 0.5, P(M|G) = 0.4, P(C|G) = 0.1 \)
- \( P(N|H) = 0.6, P(M|H) = 0.2, P(C|H) = 0.2 \)
- Maximize expected utility:
  - \( U(N) = 20, U(M) = 50, U(G) = 100 \)
  - \( EU_G = 0.5(20) + 0.4(50) + 0.1(100) = 40 \)
  - \( EU_H = 0.6(20) + 0.2(50) + 0.2(100) = 42 \)
  - Hollywood wins!

Utility of Money

- How much happier are you with an extra $1M?
- How much happier is Bill Gates with an extra $1M?
- Some have proposed:

A Sigmoidal Utility Function

\[
U(X) = \frac{100}{1 + e^{-0.0001X}}
\]
Utility & Gambling

- Suppose \( U(X) = X \), would you spend $1 for a 1 in a million chance of winning $1M?
- Suppose you start with c dollars:
  - \( EU(\text{gamble}) = \frac{1}{1000000}(1000000 - c) + (1 - \frac{1}{1000000})c = c \)
  - \( EU(\text{do nothing}) = c \)
- Starting amount doesn’t matter
- You have no expected benefit from gambling

Sigmoidal Utility & Gambling

- Suppose:
  \( U(X) = \frac{1}{1 + 2^{-0.00001X}} \)
- Suppose you start with $1M
  - \( EU(\text{gamble}) - EU(\text{do nothing}) = -5.7 \times 10^{-7} \)
  - Winning is worthless
- Suppose you start with -$1M
  - \( EU(\text{gamble}) - EU(\text{do nothing}) = +4.9 \times 10^{-6} \)
  - Gambling is rational because losing doesn’t hurt

Additive Independence

- Suppose it makes me happy to have my car clean
- Suppose it makes me happy to have coffee
- \( U = U(\text{coffee}) + U(\text{clean}) \)
- It seems that these don’t interact
- However, suppose there’s a tea variable
- \( U = U(\text{coffee}) + U(\text{tea}) + U(\text{clean}) \) ???
- Probably not. I’d need \( U(\text{coffee, tea}) + U(\text{clean}) \)
- Often implicit!

Value of Information

- Expected utility of action a with evidence E:
  \( EU_a(A | E) = \max_a \sum_i P(S_i | E, a)U(S_i) \)
- Expected utility given new evidence \( E' \)
  \( EU_{a,E'}(A | E, E') = \max_a \sum_i P(S_i | E, E', a)U(S_i) \)
- Value of knowing \( E' \) (value of perfect information)
  \( VPI_{a,E'}(E) = \left[ \sum_i P(E | E')EU_{a,E'}(a | E', E) \right] - EU(a | E) \)

- Expected utility given (weighted)
- New information
- Previous
- Expected utility

Properties of VOI

- VOI is non-negative!
- VOI is order independent
- VOI is not additive
- VOI is easy to compute and is often used to determine how much you should pay for one extra piece of information. Why is this myopic?

For example, knowing X AND Y together may be useful, while knowing just one alone may be useless.

More Properties of VOI

- Acquiring information optimally is very difficult
- Need to construct a conditional plan for every possible outcome before you ask for even the first piece of information
- Suppose you’re a doctor planning to treat a patient
- Picking the optimal test to do first requires that you consider all subsequent tests and all possible treatments as a result of these tests
- General versions of this problem are intractable!
Decision Theory as Search

- Can view DT probs as search probs
- States = atomic events

![Decision Theory Diagram](image)

DT as Search

- Attach costs to arcs, leaves
- Path with lowest expected cost = optimal
- Minimizing expected cost = maximizing expected utility
- Expectimax

\[
V(n_{\text{max}}) = \max_{s_{\text{successor}(n)}} V(s)
\]

\[
V(n_{\text{chance}}) = \sum_{s_{\text{successor}(s)}} V(s)p(s)
\]

The Form of DT Solutions

- The solution to a DT problem with many steps isn’t linear in the number of steps. (Why?)
- What does this say about computational costs?
- What does this say about the hope for exploiting heuristics?

Conclusions

- Decision theory provides a framework for optimal decision making
- Principle: Maximize Expected Utility
- Easy to describe in principle
- Application to complex problems can require advanced planning and probabilistic reasoning techniques