CS 296.1
Mathematical Modelling of Continuous Systems

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Chapter 6

Ordinary Differential Systems

In this chapter we use the theory developed in chapter 5 in order to solve systems of first-order linear differential equations with constant coefficients. These systems have the following form:

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}(t) \tag{6.1}$$

$$\mathbf{x}(0) = \mathbf{x}_0 \tag{6.2}$$

where $\mathbf{x} = \mathbf{x}(t)$ is an n-dimensional vector function of time t, the dot denotes differentiation, the coefficients a_{ij} in the $n \times n$ matrix A are constant, and the vector function $\mathbf{b}(t)$ is a function of time. The equation (6.2), in which $\mathbf{x}_0$ is a known vector, defines the *initial value* of the solution.

First, we show that *scalar* differential equations of order greater than one can be reduced to *systems* of first-order differential equations. Then, in section 6.2, we recall a general result for the solution of first-order differential systems from the elementary theory of differential equations. In section 6.3, we make this result more specific by showing that the solution to a homogeneous system is a linear combination of exponentials multiplied by polynomials in t. This result is based on the Schur decomposition introduced in chapter 5, which is numerically preferable to the more commonly used Jordan canonical form. Finally, in sections 6.4 and 6.5, we set up and solve a particular differential system as an illustrative example.

6.1 Scalar Differential Equations of Order Higher than One

The first-order system (6.1) subsumes also the case of a scalar differential equation of order n, possibly greater than 1,

$$\frac{d^n y}{dt^n} + c_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots + c_1\frac{dy}{dt} + c_0 y = b(t) \; . \tag{6.3}$$

In fact, such an equation can be reduced to a first-order system of the form (6.1) by introducing the n-dimensional vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y \\ \frac{dy}{dt} \\ \vdots \\ \frac{d^{n-1}y}{dt^{n-1}} \end{bmatrix} .$$

With this definition, we have

$$\begin{aligned} \frac{d^i y}{dt^i} &= x_{i+1} \qquad \text{for } i = 0, \ldots, n-1 \\ \frac{d^n y}{dt^n} &= \frac{dx_n}{dt} \; , \end{aligned}$$

and $\mathbf{x}$ satisfies the additional $n-1$ equations

$$x_{i+1} = \frac{dx_i}{dt} \tag{6.4}$$

for $i = 1, \ldots, n-1$. If we write the original system (6.3) together with the $n-1$ differential equations (6.4), we obtain the first-order system

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}(t)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -c_0 & -c_1 & -c_2 & \cdots & -c_{n-1} \end{bmatrix}$$

is the so-called *companion matrix* of (6.3) and

$$\mathbf{b}(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b(t) \end{bmatrix}.$$

6.2 General Solution of a Linear Differential System

We know from the general theory of differential equations that a general solution of system (6.1) with initial condition (6.2) is given by

$$\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t)$$

where $\mathbf{x}_h(t)$ is the solution of the homogeneous system

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} \\ \mathbf{x}(0) &= \mathbf{x}_0 \end{aligned}$$

and $\mathbf{x}_p(t)$ is a particular solution of

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + \mathbf{b}(t) \\ \mathbf{x}(0) &= \mathbf{0}\,. \end{aligned}$$

The two solution components $\mathbf{x}_h$ and $\mathbf{x}_p$ can be written by means of the *matrix exponential*, introduced in the following.

For the scalar exponential $e^{\lambda t}$ we can write a Taylor series expansion

$$e^{\lambda t} = 1 + \frac{\lambda t}{1!} + \frac{\lambda^2 t^2}{2!} + \cdots = \sum_{j=0}^{\infty} \frac{\lambda^j t^j}{j!}.$$

Usually[1], in calculus classes, the exponential is introduced by other means, and the Taylor series expansion above is proven as a property.

For matrices, the exponential e^Z of a matrix $Z \in \mathbf{R}^{n\times n}$ is instead *defined* by the infinite series expansion

$$e^Z = I + \frac{Z}{1!} + \frac{Z^2}{2!} + \cdots = \sum_{j=0}^{\infty} \frac{Z^j}{j!}.$$

[1]Not always. In some treatments, the exponential is *defined* through its Taylor series.

Here I is the $n \times n$ identity matrix, and the general term $Z^j/j!$ is simply the matrix Z raised to the jth power divided by the scalar $j!$. It turns out that this infinite sum converges (to an $n \times n$ matrix which we write as e^Z) for every matrix Z. Substituting $Z = At$ gives

$$e^{At} = I + \frac{At}{1!} + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots = \sum_{j=0}^{\infty} \frac{A^jt^j}{j!}. \tag{6.5}$$

Differentiating both sides of (6.5) gives

$$\begin{aligned} \frac{de^{At}}{dt} &= A + \frac{A^2t}{1!} + \frac{A^3t^2}{2!} + \cdots \\ &= A\left(I + \frac{At}{1!} + \frac{A^2t^2}{2!} + \cdots\right) \\ \frac{de^{At}}{dt} &= Ae^{At}. \end{aligned}$$

Thus, for any vector $\mathbf{w}$, the function $\mathbf{x}_h(t) = e^{At}\mathbf{w}$ satisfies the homogeneous differential system

$$\dot{\mathbf{x}}_h = A\mathbf{x}_h .$$

By using the initial values (6.2) we obtain $\mathbf{w} = \mathbf{x}_0$, and

$$\mathbf{x}_h(t) = e^{At}\mathbf{x}(0) \tag{6.6}$$

is a solution to the differential system (6.1) with $\mathbf{b}(t) = \mathbf{0}$ and initial values (6.2). It can be shown that this solution is unique.

From the elementary theory of differential equations, we also know that a particular solution to the nonhomogeneous ($\mathbf{b}(t) \neq \mathbf{0}$) equation (6.1) is given by

$$\mathbf{x}_p(t) = \int_0^t e^{A(t-s)}\,\mathbf{b}(s)\,ds .$$

This is easily verified, since by differentiating this expression for $\mathbf{x}_p$ we obtain

$$\dot{\mathbf{x}}_p = Ae^{At}\int_0^t e^{-As}\,\mathbf{b}(s)\,ds + e^{At}\,e^{-At}\,\mathbf{b}(t) = A\mathbf{x}_p + \mathbf{b}(t) ,$$

so $\mathbf{x}_p$ satisfies equation (6.1).

In summary, we have the following result.

The solution to

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}(t) \tag{6.7}$$

with initial value

$$\mathbf{x}(0) = \mathbf{x}_0 \tag{6.8}$$

is

$$\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t) \tag{6.9}$$

where

$$\mathbf{x}_h(t) = e^{At}\mathbf{x}(0) \tag{6.10}$$

and

$$\mathbf{x}_p(t) = \int_0^t e^{A(t-s)}\,\mathbf{b}(s)\,ds . \tag{6.11}$$

Since we now have a formula for the general solution to a linear differential system, we seem to have all we need. However, we do not know how to compute the matrix exponential. The naive solution to use the definition (6.5) requires too many terms for a good approximation. As we have done for the SVD and the Schur decomposition, we will only point out that several methods exist for computing a matrix exponential, but we will not discuss how this is done[2]. In a fundamental paper on the subject, *Nineteen dubious ways to compute the exponential of a matrix* (SIAM Review, vol. 20, no. 4, pp. 801-36), Cleve Moler and Charles Van Loan discuss a large number of different methods, pointing out that no one of them is appropriate for all situations. A full discussion of this matter is beyond the scope of these notes.

When the matrix A is constant, as we currently assume, we can be much more specific about the structure of the solution (6.9) of system (6.7), and particularly so about the solution $\mathbf{x}_h(t)$ to the homogeneous part. Specifically, the matrix exponential (6.10) can be written as a linear combination, with constant vector coefficients, of scalar exponentials multiplied by polynomials. In the general theory of linear differential systems, this is shown via the Jordan canonical form. However, in the paper cited above, Moler and Van Loan point out that the Jordan form cannot be computed reliably, and small perturbations in the data can change the results dramatically. Fortunately, a similar result can be found through the Schur decomposition introduced in chapter 5. The next section shows how to do this.

6.3 Structure of the Solution

For the homogeneous case $\mathbf{b}(t) = \mathbf{0}$, consider the first order system of linear differential equations

$$\dot{\mathbf{x}} = A\mathbf{x} \tag{6.12}$$

$$\mathbf{x}(0) = \mathbf{x}_0 \, . \tag{6.13}$$

Two cases arise: either A admits n distinct eigenvalues, or is does not. In chapter 5, we have seen that if (but not only if) A has n distinct eigenvalues then it has n linearly independent eigenvectors (theorem 5.1.1), and we have shown how to find $\mathbf{x}_h(t)$ by solving an eigenvalue problem. In section 6.3.1, we briefly review this solution. Then, in section 6.3.2, we show how to compute the homogeneous solution $\mathbf{x}_h(t)$ in the extreme case of an $n \times n$ matrix A with n *coincident* eigenvalues.

To be sure, we have seen that matrices with coincident eigenvalues can still have a full set of linearly independent eigenvectors (see for instance the identity matrix). However, the solution procedure we introduce in section 6.3.2 for the case of n coincident eigenvalues can be applied regardless to how many linearly independent eigenvectors exist. If the matrix has a full complement of eigenvectors, the solution obtained in section 6.3.2 is the same as would be obtained with the method of section 6.3.1.

Once these two extreme cases (nondefective matrix or all-coincident eigenvalues) have been handled, we show a general procedure in section 6.3.3 for solving a homogeneous or nonhomogeneous differential system for any, square, constant matrix A, defective or not. This procedure is based on backsubstitution, and produces a result analogous to that obtained via Jordan decomposition for the homogeneous part $\mathbf{x}_h(t)$ of the solution. However, since it is based on the numerically sound Schur decomposition, the method of section 6.3.3 is superior in practice. For a nonhomogeneous system, the procedure can be carried out analytically if the functions in the right-hand side vector $\mathbf{b}(t)$ can be integrated.

6.3.1 A is Not Defective

In chapter 5 we saw how to find the homogeneous part $\mathbf{x}_h(t)$ of the solution when A has a full set of n linearly independent eigenvectors. This result is briefly reviewed in this section for convenience.[3]

If A is not defective, then it has n linearly independent eigenvectors $\mathbf{q}_1, \ldots, \mathbf{q}_n$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. Let

$$Q = \begin{bmatrix} \mathbf{q}_1 & \cdots & \mathbf{q}_n \end{bmatrix} .$$

[2] In Matlab, `expm(A)` is the matrix exponential of `A`.

[3] Parts of this subsection and of the following one are based on notes written by Scott Cohen.

This square matrix is invertible because its columns are linearly independent. Since $A\mathbf{q}_i = \lambda_i \mathbf{q}_i$, we have

$$AQ = Q\Lambda, \tag{6.14}$$

where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ is a square diagonal matrix with the eigenvalues of A on its diagonal. Multiplying both sides of (6.14) by Q^{-1} on the right, we obtain

$$A = Q\Lambda Q^{-1}. \tag{6.15}$$

Then, system (6.12) can be rewritten as follows:

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} \\ \dot{\mathbf{x}} &= Q\Lambda Q^{-1}\mathbf{x} \\ Q^{-1}\dot{\mathbf{x}} &= \Lambda Q^{-1}\mathbf{x} \\ \dot{\mathbf{y}} &= \Lambda \mathbf{y}, \end{aligned} \tag{6.16}$$

where $\mathbf{y} = Q^{-1}\mathbf{x}$. The last equation (6.16) represents n uncoupled, homogeneous, differential equations $\dot{y}_i = \lambda_i y_i$. The solution is

$$\mathbf{y}_h(t) = e^{\Lambda t}\mathbf{y}(0),$$

where

$$e^{\Lambda t} = \text{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_n t}).$$

Using the relation $\mathbf{x} = Q\mathbf{y}$, and the consequent relation $\mathbf{y}(0) = Q^{-1}\mathbf{x}(0)$, we see that the solution to the homogeneous system (6.12) is

$$\mathbf{x}_h(t) = Qe^{\Lambda t}Q^{-1}\mathbf{x}(0).$$

If A is normal, that is, if it has n orthonormal eigenvectors $q_1, \ldots q_n$, then Q is replaced by the Hermitian matrix $S = \begin{bmatrix} \mathbf{s}_1 & \cdots & \mathbf{s}_n \end{bmatrix}$, Q^{-1} is replaced by S^H, and the solution to (6.12) becomes

$$\mathbf{x}_h(t) = Se^{\Lambda t}S^H\mathbf{x}(0).$$

6.3.2 A Has n Coincident Eigenvalues

When $A = Q\Lambda Q^{-1}$, we derived that the solution to (6.12) is $\mathbf{x}_h(t) = Qe^{\Lambda t}Q^{-1}\mathbf{x}(0)$. Comparing with (6.6), it should be the case that

$$e^{Q(\Lambda t)Q^{-1}} = Qe^{\Lambda t}Q^{-1}.$$

This follows easily from the definition of e^Z and the fact that $(Q(\Lambda t)Q^{-1})^j = Q(\Lambda t)^j Q^{-1}$. Similarly, if $A = S\Lambda S^H$, where S is Hermitian, then the solution to (6.12) is $\mathbf{x}_h(t) = Se^{\Lambda t}S^H\mathbf{x}(0)$, and

$$e^{S(\Lambda t)S^H} = Se^{\Lambda t}S^H.$$

How can we compute the matrix exponential in the extreme case in which A has n coincident eigenvalues, regardless of the number of its linearly independent eigenvectors? In any case, A admits a Schur decomposition

$$A = STS^H$$

(theorem 5.3.2). We recall that S is a unitary matrix and T is upper triangular with the eigenvalues of A on its diagonal. Thus we can write T as

$$T = \Lambda + N,$$

where Λ is diagonal and N is *strictly* upper triangular. The solution (6.6) in this case becomes

$$\mathbf{x}_h(t) = e^{S(Tt)S^H}\mathbf{x}(0) = Se^{Tt}S^H\mathbf{x}(0) = Se^{\Lambda t + Nt}S^H\mathbf{x}(0).$$

Thus we can compute (6.6) if we can compute $e^{Tt} = e^{\Lambda t + Nt}$. This turns out to be almost as easy as computing $e^{\Lambda t}$ when the diagonal matrix Λ is a multiple of the identity matrix:

$$\Lambda = \lambda I$$

that is, when all the eigenvalues of A coincide. In fact, in this case, Λt and Nt commute:

$$\Lambda t \, Nt = \lambda \, It \, Nt = \lambda t \, Nt = Nt \, \lambda t = Nt \, \lambda \, It = Nt \, \Lambda t \, .$$

It can be shown that if two matrices Z_1 and Z_2 commute, that is if

$$Z_1 Z_2 = Z_2 Z_1 \, ,$$

then

$$e^{Z_1+Z_2} = e^{Z_1} e^{Z_2} = e^{Z_2} e^{Z_1} \, .$$

Thus, in our case, we can write

$$e^{\Lambda t + Nt} = e^{\Lambda t} e^{Nt}.$$

We already know how to compute $e^{\Lambda t}$, so it remains to show how to compute e^{Nt}. The fact that Nt is strictly upper triangular makes the computation of this matrix exponential much simpler than for a general matrix Z.

Suppose, for example, that N is 4×4. Then N has three nonzero superdiagonals, N^2 has two nonzero superdiagonals, N^3 has one nonzero superdiagonal, and N^4 is the zero matrix:

$$\begin{aligned} N &= \begin{bmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow N^2 = \begin{bmatrix} 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \\ N^3 &= \begin{bmatrix} 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow N^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} . \end{aligned}$$

In general, for a strictly upper triangular $n \times n$ matrix, we have $N^j = 0$ for all $j \geq n$ (i.e., N is nilpotent of order n). Therefore,

$$e^{Nt} = \sum_{j=0}^{\infty} \frac{N^j t^j}{j!} = \sum_{j=0}^{n-1} \frac{N^j t^j}{j!}$$

is simply a finite sum, and the exponential reduces to a matrix polynomial.

In summary, the general solution to the homogeneous differential system (6.12) with initial value (6.13) when the $n \times n$ matrix A has n coincident eigenvalues is given by

$$\mathbf{x}_h(t) = S e^{\Lambda t} \sum_{j=0}^{n-1} \frac{N^j t^j}{j!} S^H \mathbf{x}_0 \tag{6.17}$$

where

$$A = S(\Lambda + N) S^H$$

is the Schur decomposition of A,

$$\Lambda = \lambda I$$

is a multiple of the identity matrix containing the coincident eigenvalues of A on its diagonal, and N is strictly upper triangular.

6.3.3 The General Case

We are now ready to solve the linear differential system

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + \mathbf{b}(t) && (6.18)\\ \mathbf{x}(0) &= \mathbf{x}_0 && (6.19) \end{aligned}$$

in the general case of a constant matrix A, defective or not, with arbitrary $\mathbf{b}(t)$. In fact, let $A = STS^H$ be the Schur decomposition of A, and consider the transformed system

$$\dot{\mathbf{y}}(t) = T\mathbf{y}(t) + \mathbf{c}(t) \tag{6.20}$$

where

$$\mathbf{y}(t) = S^H\mathbf{x}(t) \quad \text{and} \quad \mathbf{c}(t) = S^H\mathbf{b}(t)\ . \tag{6.21}$$

The triangular matrix T can always be written in the following form:

$$T = \begin{bmatrix} T_{11} & \cdots & \cdots & T_{1k} \\ 0 & T_{22} & \cdots & T_{2k} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & T_{kk} \end{bmatrix}$$

where the diagonal blocks T_{ii} for $i = 1, \ldots, k$ are of size $n_i \times n_i$ (possibly 1×1) and contain all-coincident eigenvalues. The remaining nonzero blocks T_{ij} with $i < j$ can be in turn bundled into matrices

$$R_i = \begin{bmatrix} T_{i,i+1} & \cdots & T_{i,k} \end{bmatrix}$$

that contain everything to the right of the corresponding T_{ii}. The vector $\mathbf{c}(t)$ can be partitioned correspondingly as follows

$$\mathbf{c}(t) = \begin{bmatrix} \mathbf{c}_1(t) \\ \vdots \\ \mathbf{c}_k(t) \end{bmatrix}$$

where $\mathbf{c}_i$ has n_i entries, and the same can be done for

$$\mathbf{y}(t) = \begin{bmatrix} \mathbf{y}_1(t) \\ \vdots \\ \mathbf{y}_k(t) \end{bmatrix}$$

and for the initial values

$$\mathbf{y}(0) = \begin{bmatrix} \mathbf{y}_1(0) \\ \vdots \\ \mathbf{y}_k(0) \end{bmatrix} .$$

The triangular system (6.20) can then be solved by backsubstitution as follows:

for $i = k$ down to 1
    if $i < k$
        $\mathbf{d}_i(t) = R_i[\mathbf{y}_{i+1}(t), \ldots, \mathbf{y}_k(t)]^T$
    else
        $\mathbf{d}_i(t) = \mathbf{0}$ (an n_k-dimensional vector of zeros)
    end
    $T_{ii} = \lambda_i\, I + N_i$ (diagonal and strictly upper-triangular part of T_{ii})
    $\mathbf{y}_i(t) = e^{\lambda_i I t} \sum_{j=0}^{n_i-1} \frac{N_i^j t^j}{j!} \mathbf{y}_i(0) + \int_0^t \left(e^{\lambda_i I(t-s)} \sum_{j=0}^{n_i-1} \frac{N_i^j (t-s)^j}{j!} \right) (\mathbf{c}_i(s) + \mathbf{d}_i(s))\, ds$
end.

In this procedure, the expression for $\mathbf{y}_i(t)$ is a direct application of equations (6.9), (6.10), (6.11), and (6.17) with $S = I$. In the general case, the applicability of this routine depends on whether the integral in the expression for $\mathbf{y}_i(t)$ can be computed analytically. This is certainly the case when $\mathbf{b}(t)$ is a constant vector $\mathbf{b}$, because then the integrand is a linear combination of exponentials multiplied by polynomials in $t - s$, which can be integrated by parts.

The solution $\mathbf{x}(t)$ for the original system (6.18) is then

$$\mathbf{x}(t) = S\mathbf{y}(t) .$$

As an illustration, we consider a very small example, the 2×2 homogeneous, triangular case,

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ 0 & t_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} . \tag{6.22}$$

When $t_{11} = t_{22} = \lambda$, we obtain

$$\mathbf{y}(t) = e^{\lambda I t} \begin{bmatrix} 1 & t_{12}t \\ 0 & 1 \end{bmatrix} \mathbf{y}(0) .$$

In scalar form, this becomes

$$\begin{aligned} y_1(t) &= (y_1(0) + t_{12}y_2(0)\,t)\,e^{\lambda t} \\ y_2(t) &= y_2(0)\,e^{\lambda t} , \end{aligned}$$

and it is easy to verify that this solution satisfies the differential system (6.22).

When $t_{11} = \lambda_1 \neq t_{22} = \lambda_2$, we could solve the system by finding the eigenvectors of T, since we know that in this case two linearly independent eigenvectors exist (theorem 5.1.1). Instead, we apply the backsubstitution procedure introduced in this section. The second equation of the system,

$$\dot{y}_2(t) = t_{22}y_2$$

has solution

$$y_2(t) = y_2(0)\,e^{\lambda_2 t} .$$

We then have

$$d_1(t) = t_{12}y_2(t) = t_{12}y_2(0)\,e^{\lambda_2 t}$$

and

$$\begin{aligned} y_1(t) &= y_1(0)e^{\lambda_1 t} + \int_0^t e^{\lambda_1(t-s)} d_1(s)\,ds \\ &= y_1(0)e^{\lambda_1 t} + t_{12}y_2(0)\,e^{\lambda_1 t} \int_0^t e^{-\lambda_1 s} e^{\lambda_2 s}\,ds \\ &= y_1(0)e^{\lambda_1 t} + t_{12}y_2(0)\,e^{\lambda_1 t} \int_0^t e^{(\lambda_2 - \lambda_1)s}\,ds \\ &= y_1(0)e^{\lambda_1 t} + \frac{t_{12}y_2(0)}{\lambda_2 - \lambda_1}\,e^{\lambda_1 t}(e^{(\lambda_2-\lambda_1)t} - 1) \\ &= y_1(0)e^{\lambda_1 t} + \frac{t_{12}y_2(0)}{\lambda_2 - \lambda_1}\,(e^{\lambda_2 t} - e^{\lambda_1 t}) \end{aligned}$$

Exercise: verify that this solution satisfies both the differential equation (6.22) and the initial value equation $\mathbf{y}(0) = \mathbf{y}_0$.

Thus, the solutions to system (6.22) for $t_{11} = t_{22}$ and for $t_{11} \neq t_{22}$ have different forms. While $y_2(t)$ is the same in both cases, we have

$$\begin{aligned} y_1(t) = y_1(0)\,e^{\lambda t} + t_{12}y_2(0)\,t\,e^{\lambda t} \quad &\text{if} \quad t_{11} = t_{22} \\ y_1(t) = y_1(0)e^{\lambda_1 t} + \frac{t_{12}y_2(0)}{\lambda_2 - \lambda_1}\,(e^{\lambda_2 t} - e^{\lambda_1 t}) \quad &\text{if} \quad t_{11} \neq t_{22} . \end{aligned}$$

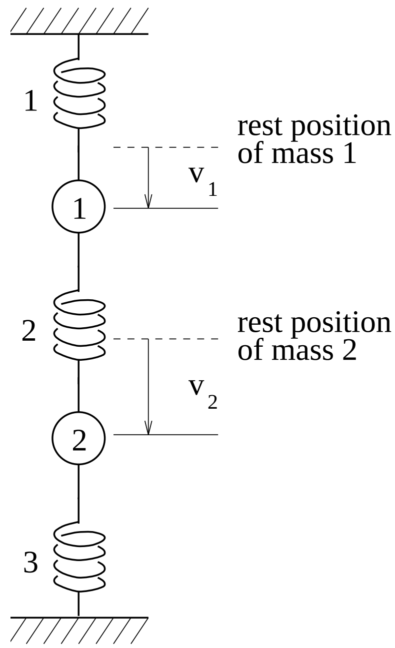

Figure 6.1: A system of masses and springs. In the absence of external forces, the two masses would assume the positions indicated by the dashed lines.

This would seem to present numerical difficulties when $t_{11} \approx t_{22}$, because the solution would suddenly switch from one form to the other as the difference between t_{11} and t_{22} changes from about zero to exactly zero or viceversa. This, however, is not a problem. In fact,

$$\lim_{\lambda_1 \to \lambda} \frac{e^{\lambda t} - e^{\lambda_1 t}}{\lambda - \lambda_1} = t\, e^{\lambda t} \ ,$$

and the transition between the two cases is smooth.

6.4 A Concrete Example

In this section we set up and solve a more concrete example of a system of differential equations. The initial system has two second-order equations, and is transformed into a first-order system with four equations. The 4×4 matrix of the resulting system has an interesting structure, which allows finding eigenvalues and eigenvectors analytically with a little trick. The point of this section is to show how to transform the complex formal solution of the differential system, computed with any of the methods described above, into a real solution in a form appropriate to the problem at hand.

Consider the mechanical system in figure 6.1. Suppose that we want to study the evolution of the system over time. Since forces are proportional to accelerations, because of Newton's law, and since accelerations are second derivatives of position, the new equations are differential. Because differentiation occurs only with respect to one variable, time, these are *ordinary* differential equations, as opposed to partial.

In the following we write the differential equations that describe this system. Two linear differential equations of the second order[4] result. We will then transform these into four linear differential equations of the first order.

[4]Recall that the order of a differential equation is the highest degree of derivative that appears in it.

By Hooke's law, the three springs exert forces that are proportional to the springs' elongations:

$$\begin{aligned} f_1 &= c_1 v_1 \\ f_2 &= c_2(v_2 - v_1) \\ f_3 &= -c_3 v_2 \end{aligned}$$

where the c_i are the positive spring constants (in newtons per meter).

The accelerations of masses 1 and 2 (springs are assumed to be massless) are proportional to their accelerations, according to Newton's second law:

$$\begin{aligned} m_1 \ddot{v}_1 &= -f_1 + f_2 = -c_1 v_1 + c_2(v_2 - v_1) = -(c_1 + c_2)v_1 + c_2 v_2 \\ m_2 \ddot{v}_2 &= -f_2 + f_3 = -c_2(v_2 - v_1) - c_3 v_2 = c_2 v_1 - (c_2 + c_3)v_2 \end{aligned}$$

or, in matrix form,

$$\ddot{\mathbf{v}} = B\mathbf{v} \tag{6.23}$$

where

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -\frac{c_1+c_2}{m_1} & \frac{c_2}{m_1} \\ \frac{c_2}{m_2} & -\frac{c_2+c_3}{m_2} \end{bmatrix} .$$

We also assume that initial conditions

$$\mathbf{v}(0) \quad \text{and} \quad \dot{\mathbf{v}}(0) \tag{6.24}$$

are given, which specify positions and velocities of the two masses at time $t = 0$.

To solve the second-order system (6.23), we will first transform it to a system of four first-order equations. As shown in the introduction to this chapter, the trick is to introduce variables to denote the first-order derivatives of $\mathbf{v}$, so that second-order derivatives of $\mathbf{v}$ are first-order derivatives of the new variables. For uniformity, we define four new variables

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} \tag{6.25}$$

so that

$$u_3 = \dot{v}_1 \quad \text{and} \quad u_4 = \dot{v}_2 ,$$

while the original system (6.23) becomes

$$\begin{bmatrix} \dot{u_3} \\ \dot{u_4} \end{bmatrix} = B \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} .$$

We can now gather these four first-order differential equations into a single system as follows:

$$\dot{\mathbf{u}} = A\mathbf{u} \tag{6.26}$$

where

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \multicolumn{2}{c}{\multirow{2}{*}{B}} & 0 & 0 \\ & & 0 & 0 \end{bmatrix} .$$

Likewise, the initial conditions (6.24) are replaced by the (known) vector

$$\mathbf{u}(0) = \begin{bmatrix} \mathbf{v}(0) \\ \dot{\mathbf{v}}(0) \end{bmatrix} .$$

In the next section we solve equation (6.26).

6.5 Solution of the Example

Not all matrices have a full set of linearly independent eigenvectors. With the system of springs in figure 6.1, however, we are lucky. The eigenvalues of A are solutions to the equation

$$A\mathbf{x} = \lambda \mathbf{x} \ , \tag{6.27}$$

where we recall that

$$A = \begin{bmatrix} 0 & I \\ B & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -\frac{c_1+c_2}{m_1} & \frac{c_2}{m_1} \\ \frac{c_2}{m_2} & -\frac{c_2+c_3}{m_2} \end{bmatrix} .$$

Here, the zeros in A are 2×2 matrices of zeros, and I is the 2×2 identity matrix. If we partition the vector $\mathbf{x}$ into its upper and lower halves $\mathbf{y}$ and $\mathbf{z}$,

$$\mathbf{x} = \begin{bmatrix} \mathbf{y} \\ \mathbf{z} \end{bmatrix} ,$$

we can write

$$A\mathbf{x} = \begin{bmatrix} 0 & I \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{z} \\ B\mathbf{y} \end{bmatrix}$$

so that the eigenvalue equation (6.27) can be written as the following pair of equations:

$$\begin{aligned} \mathbf{z} &= \lambda \mathbf{y} \\ B\mathbf{y} &= \lambda \mathbf{z} \ , \end{aligned} \tag{6.28}$$

which yields

$$B\mathbf{y} = \mu \mathbf{y} \quad \text{with} \quad \mu = \lambda^2 \ .$$

In other words, the eigenvalues of A are the square roots of the eigenvalues of B: if we denote the two eigenvalues of B as μ_1 and μ_2, then the eigenvalues of A are

$$\lambda_1 = \sqrt{\mu_1} \qquad \lambda_2 = -\sqrt{\mu_1} \qquad \lambda_3 = \sqrt{\mu_2} \qquad \lambda_4 = -\sqrt{\mu_2} \ .$$

The eigenvalues μ_1 and μ_2 of B are the solutions of

$$\det(B - \mu I) = \left(\frac{c_1 + c_2}{m_1} + \mu \right) \left(\frac{c_2 + c_3}{m_2} + \mu \right) - \frac{c_2^2}{m_1 m_2} = \mu^2 + 2\alpha\mu + \beta = 0$$

where

$$\alpha = \frac{1}{2} \left(\frac{c_1 + c_2}{m_1} + \frac{c_2 + c_3}{m_2} \right) \quad \text{and} \quad \beta = \frac{c_1 c_2 + c_1 c_3 + c_2 c_3}{m_1 m_2}$$

are positive constants that depend on the elastic properties of the springs and on the masses. We then obtain

$$\mu_{1,2} = -\alpha \pm \gamma \ ,$$

where

$$\gamma = \sqrt{\alpha^2 - \beta} = \sqrt{\frac{1}{4} \left(\frac{c_1 + c_2}{m_1} - \frac{c_2 + c_3}{m_2} \right)^2 + \frac{c_2^2}{m_1 m_2}} \ .$$

The constant γ is real because the radicand is nonnegative. We also have that $\alpha \geq \gamma$, so that the two solutions $\mu_{1,2}$ are real and negative, and the four eigenvalues of A,

$$\lambda_1 = \sqrt{-\alpha + \gamma} \, , \quad \lambda_2 = -\sqrt{-\alpha + \gamma} \, , \tag{6.29}$$

$$\lambda_3 = \sqrt{-\alpha - \gamma} \, , \quad \lambda_4 = -\sqrt{-\alpha - \gamma} \tag{6.30}$$

come in nonreal, complex-conjugate pairs. This is to be expected, since our system of springs obviously exhibits an oscillatory behavior.

Also the eigenvectors of A can be derived from those of B. In fact, from equation (6.28) we see that if $\mathbf{y}$ is an eigenvector of B corresponding to eigenvalue $\mu = \lambda^2$, then there are two corresponding eigenvectors for A of the form

$$\mathbf{x} = \begin{bmatrix} \mathbf{y} \\ \pm\lambda\mathbf{y} \end{bmatrix} . \tag{6.31}$$

The eigenvectors of B are the solutions of

$$(B - (-\alpha \pm \gamma)I)\mathbf{y} = 0 . \tag{6.32}$$

Since $\pm(-\alpha \pm \gamma)$ are eigenvalues of B, the determinant of this equation is zero, and the two scalar equations in (6.32) must be linearly dependent. The first equation reads

$$-\left(\frac{c_1 + c_2}{m_1} - \alpha \pm \gamma\right) y_1 + \frac{c_2}{m_1} y_2 = 0$$

and is obviously satisfied by any vector of the form

$$\mathbf{y} = k \begin{bmatrix} \frac{c_2}{m_1} \\ \frac{c_1+c_2}{m_1} - \alpha \pm \gamma \end{bmatrix}$$

where k is an arbitrary constant. For $k \neq 0$, $\mathbf{y}$ denotes the two eigenvectors of B, and from equation (6.31) the four eigenvectors of A are proportional to the four columns of the following matrix:

$$Q = \begin{bmatrix} \frac{c_2}{m_1} & \frac{c_2}{m_1} & \frac{c_2}{m_1} & \frac{c_2}{m_1} \\ a + \lambda_1^2 & a + \lambda_2^2 & a + \lambda_3^2 & a + \lambda_4^2 \\ \lambda_1 \frac{c_2}{m_1} & \lambda_2 \frac{c_2}{m_1} & \lambda_3 \frac{c_2}{m_1} & \lambda_4 \frac{c_2}{m_1} \\ \lambda_1 \left(a + \lambda_1^2\right) & \lambda_2 \left(a + \lambda_2^2\right) & \lambda_3 \left(a + \lambda_3^2\right) & \lambda_4 \left(a + \lambda_4^2\right) \end{bmatrix} \tag{6.33}$$

where

$$a = \frac{c_1 + c_2}{m_1} .$$

The general solution to the first-order differential system (6.26) is then given by equation (6.17). Since we just found four distinct eigenvectors, however, we can write more simply

$$\mathbf{u}(t) = Q e^{\Lambda t} Q^{-1} \mathbf{u}(0) \tag{6.34}$$

where

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} .$$

In these expressions, the values of λ_i are given in equations (6.30), and Q is in equation (6.33).

Finally, the solution to the original, second-order system (6.23) can be obtained from equation (6.25) by noticing that $\mathbf{v}$ is equal to the first two components of $\mathbf{u}$.

This completes the solution of our system of differential equations. However, it may be useful to add some algebraic manipulation in order to show that the solution is indeed oscillatory. As we see in the following, the masses' motions can be described by the superposition of two sinusoids whose frequencies depend on the physical constants involved (masses and spring constants). The amplitudes and phases of the sinusoids, on the other hand, depend on the initial conditions.

To simplify our manipulation, we note that

$$\mathbf{u}(t) = Q e^{\Lambda t} \mathbf{w} ,$$

where we defined

$$\mathbf{w} = Q^{-1}\mathbf{u}(0) \; . \tag{6.35}$$

We now leave the constants in $\mathbf{w}$ unspecified, and derive the general solution $\mathbf{v}(t)$ for the original, second-order problem. Numerical values for the constants can be found from the initial conditions $\mathbf{u}(0)$ by equation (6.35). We have

$$\mathbf{v}(t) = Q(1:2,:)e^{\Lambda t}\mathbf{w} \; ,$$

where $Q(1:2,:)$ denotes the first two rows of Q. Since

$$\lambda_2 = -\lambda_1 \text{ and } \quad \lambda_4 = -\lambda_3$$

(see equations (6.30)), we have

$$Q(1:2,:) = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_2 \end{bmatrix}$$

where we defined

$$\mathbf{q}_1 = \begin{bmatrix} \frac{c_2}{m_1} \\ \frac{c_1+c_2}{m_1} + \lambda_1^2 \end{bmatrix} \quad \text{and} \quad \mathbf{q}_2 = \begin{bmatrix} \frac{c_2}{m_1} \\ \frac{c_1+c_2}{m_1} + \lambda_3^2 \end{bmatrix} \; .$$

Thus, we can write

$$\mathbf{v}(t) = \mathbf{q}_1 \left(k_1 e^{\lambda_1 t} + k_2 e^{-\lambda_1 t} \right) + \mathbf{q}_2 \left(k_3 e^{\lambda_3 t} + k_4 e^{-\lambda_3 t} \right) \; .$$

Since the λs are imaginary but $\mathbf{v}(t)$ is real, the k_i must come in complex-conjugate pairs:

$$k_1 = k_2^* \quad \text{and} \quad k_3 = k_4^* \; . \tag{6.36}$$

In fact, we have

$$\mathbf{v}(0) = \mathbf{q}_1(k_1 + k_2) + \mathbf{q}_2(k_3 + k_4)$$

and from the derivative

$$\dot{\mathbf{v}}(t) = \mathbf{q}_1\lambda_1 \left(k_1 e^{\lambda_1 t} - k_2 e^{-\lambda_1 t} \right) + \mathbf{q}_2\lambda_3 \left(k_3 e^{\lambda_3 t} - k_4 e^{-\lambda_3 t} \right)$$

we obtain

$$\dot{\mathbf{v}}(0) = \mathbf{q}_1\lambda_1(k_1 - k_2) + \mathbf{q}_2\lambda_3(k_3 - k_4) \; .$$

Since the vectors $\mathbf{q}_i$ are independent (assuming that the mass c_2 is nonzero), this means that

$$\begin{array}{ll} k_1 + k_2 \text{ is real} & k_1 - k_2 \text{ is purely imaginary} \\ k_3 + k_4 \text{ is real} & k_3 - k_4 \text{ is purely imaginary} \; , \end{array}$$

from which equations (6.36) follow.

Finally, by using the relation

$$\frac{e^{jx} + e^{-jx}}{2} = \cos x \; ,$$

and simple trigonometry we obtain

$$\mathbf{v}(t) = \mathbf{q}_1 A_1 \cos(\omega_1 t + \phi_1) + \mathbf{q}_2 A_2 \cos(\omega_2 t + \phi_2)$$

where

$$\begin{aligned} \omega_1 &= \sqrt{\alpha - \gamma} = \sqrt{\frac{1}{2}(a+b) - \sqrt{\frac{1}{4}(a-b)^2 + \frac{c_2^2}{m_1 m_2}}} \\ \omega_2 &= \sqrt{\alpha + \gamma} = \sqrt{\frac{1}{2}(a+b) + \sqrt{\frac{1}{4}(a-b)^2 + \frac{c_2^2}{m_1 m_2}}} \end{aligned}$$

and

$$a = \frac{c_1 + c_2}{m_1} \quad , \quad b = \frac{c_2 + c_3}{m_2} \; .$$

Notice that these two frequencies depend only on the configuration of the system, and not on the initial conditions.

The amplitudes A_i and phases ϕ_i, on the other hand, depend on the constants k_i as follows:

$$\begin{aligned} A_1 &= 2|k_1| \; , & A_2 &= 2|k_3| \\ \phi_1 &= \arctan_2(\mathsf{Im}(k_1), \mathsf{Re}(k_1)) & \phi_2 &= \arctan_2(\mathsf{Im}(k_3), \mathsf{Re}(k_3)) \end{aligned}$$

where Re, Im denote the real and imaginary part and where the two-argument function $\arctan_2$ is defined as follows for $(x, y) \neq (0, 0)$

$$\arctan_2(y, x) = \begin{cases} \arctan(\frac{y}{x}) & \text{if } x > 0 \\ \pi + \arctan(\frac{y}{x}) & \text{if } x < 0 \\ \frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0 \end{cases}$$

and is undefined for $(x, y) = (0, 0)$. This function returns the arctangent of y/x (notice the order of the arguments) in the proper quadrant, and extends the function by continuity along the y axis.

The two constants k_1 and k_3 can be found from the given initial conditions $\mathbf{v}(0)$ and $\dot{\mathbf{v}}(0)$ from equations (6.35) and (6.25).