1. Let
\[ A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \\ 0 & 4 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 0 \\ 7 \\ 8 \end{bmatrix} \].

There are at least two interesting ways to look at the system of equations \( Ax = b \) geometrically.

(a) The view by rows sees each row of the system above as the equation of a line. If there is a solution, it’s where the lines intersect. Draw a plot with the three lines. Label your axes and the line intercepts (points where the lines meet the axes).

(b) What is the solution to the system?

(c) The view by columns sees the right-hand side vector \( b \) of the system as a linear combination of the columns \( a_1 \) and \( a_2 \) of \( A \). Draw a three-dimensional plot that illustrates that \( x_1a_1 + x_2a_2 = b \) with the values above. Label your axes in detail.

(d) What can you say of \( a_1 \), \( a_2 \), and \( b \) if a \( 3 \times 2 \) system \( Ax = b \) were to admit no solution?

2. A problem is not “trigonometric” or “non trigonometric” in itself. Rather, trigonometry is sometimes one way of looking at a problem. To see this, we solve the same problem in two ways, one trigonometric, the other algebraic.

Let the two column vectors \( a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}^T \) and \( b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}^T \) collect the Cartesian coordinates of two points on the plane (the superscript \( T \) denotes matrix transposition). Similarly, let \( 0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}^T \) be the origin.

(a) Show that the area \( A \) of a parallelogram with vertices 0, \( a \), \( b \), and \( a + b \) is \( A = \|a\|\|b\| \sin \alpha \), where \( \alpha \) is the unsigned angle between \( a \) and \( b \). Your proof may be essentially in the form of a drawing.

(b) Show that \( A = a_1b_2 - a_2b_1 \). This is called the determinant of the matrix \( \begin{bmatrix} a & b \end{bmatrix} \). You may not assume knowledge of properties of the determinant in your proof. Again, a figure helps.

[Note: It is possible, but not for the faint-hearted, to write a similar proof for the volume of parallelepipeds in three dimensions.]

3. Define a plane to be the set of all points of the form \( p = \alpha a + \beta b + c \), with \( a \), \( b \), and \( c \) three fixed vectors, and with the two real numbers \( \alpha \) and \( \beta \) varying in all possible ways. This is called the parametric form of a plane. In three dimensions, two planes either intersect along a line or are parallel (when they do both, they are said to coincide). In four dimensions, this is not always the case.

Specify \( a \), \( b \), and \( c \) for each of two planes in four-dimensional space (that is, \( a \), \( b \), and \( c \) are four-dimensional vectors) that intersect at a single point.


The surface area of a unit sphere in \( d \) dimensions is
\[
S_d = \frac{2(\sqrt{\pi})^d}{\Gamma(d/2)}
\]
and the volume of a sphere of radius \( a \) in \( d \) dimensions is
\[
V_d = \frac{S_d a^d}{d}
\]
where \( \Gamma(x) \) is the gamma function defined by
\[
\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} \, du.
\]

If the gamma function looks complicated, do not worry. All we need to know is that \( \Gamma(1) = 1 \) and \( \Gamma(3/2) = \sqrt{\pi}/2 \).

Let \( \epsilon \) be a number such that \( 0 < \epsilon < a \). An \( \epsilon \)-thick shell for a \( d \)-dimensional sphere is the part of the sphere that has radius values between \( a - \epsilon \) and \( a \). For instance, for \( d = 2 \), the shell in question is a ring with inner radius \( a - \epsilon \) and outer radius \( a \). Our intuition tells us that this shell can be made arbitrarily small by shrinking \( \epsilon \) toward zero. In many dimensions, however, this shell is surprisingly thick, as you will now show.

(a) Rewrite the expression for \( S_d \) and \( V_d \) for the cases \( d = 2 \) and \( d = 3 \), to verify that you obtain the familiar expressions for the circle and the sphere.

(b) Show that the fraction of the volume of the sphere that lies in its \( \epsilon \)-thick shell is given by
\[
f = 1 - \left(1 - \frac{\epsilon}{a}\right)^d.
\]

(c) Show that for any fixed \( \epsilon \) no matter how small, this fraction tends to 1 as \( d \to \infty \).

(d) Evaluate the ratio \( f \) numerically, with \( \epsilon/a = 0.01 \), for the cases \( d = 2 \), \( d = 10 \), and \( d = 1000 \).

You just showed that, for points that are uniformly distributed inside a sphere in \( d \) dimensions where \( d \) is large, almost all of the points are concentrated in a thin shell close to the surface.

5. A different, but related failure of our intuition in many dimensions occurs when considering the ratio between the volume of the unit sphere in \( d \) dimensions, and the volume of the \( d \) dimensional cube of side 2 that contains it. The figure below shows the case \( d = 2 \). As you see, there are four “corners,” defined as the areas that are inside the square and outside the circle.

Bishop shows that in a high dimensional space most of the volume of the cube is concentrated in the large number of corners, which themselves become very long spikes. Rather than proving this, as done in Bishop’s exercises, we will try to familiarize ourselves with the picture. The origin of space is at the center of the sphere.

(a) How many corners are there in \( d \) dimensions?

(b) What is the maximum norm (that is, the greatest distance from the origin) of a point within the unit sphere?

(c) How far from the origin is a point at one of the cube’s vertices?
(d) Let \( \mathbf{v} = [x_1, \ldots, x_d]^T \) be the vector of coordinates of a generic point in \( d \) dimensions. What must be true of \( x_1, \ldots, x_d \) for \( \mathbf{v} \) to be in the cube above?

(e) What must be true of \( x_1, \ldots, x_d \) for \( \mathbf{v} \) to be in the unit sphere?

(f) Picking a random point in the cube is the same as picking random, independent, and uniformly distributed values for \( x_1, \ldots, x_d \). Argue, in qualitative and intuitive terms only, that a random vector in the cube is much more likely to be outside the unit sphere than inside it.