Questions may continue on the back. Please write clearly. What I cannot read, I will not grade. Typed homework is preferable. A good compromise is to type the words and write the math by hand.

1. If two matrices \( A \) and \( B \) commute, that is, if \( AB = BA \), then
\[
e^{A+B} = e^A e^B.
\]
(1)

This exercise will take you through the proof of this fact.

(a) Let
\[
A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.
\]

Show that \( A \) and \( B \) do not commute, and compute the matrices \( e^A e^B \), \( e^B e^A \), and \( e^{A+B} \). This should convince you that equation (1) should not be taken for granted. Give exact answers, not numerical approximations.

(b) Prove by induction that if \( AB = BA \) then \( B \) commutes with every power of \( A \).

(c) Prove that \( B \) commutes with \( e^{A t} \) for every real number \( t \).

(d) Our main theorem is proven once we show that the matrix difference
\[
F(t) = e^{(A+B)t} - e^{At} e^{Bt}
\]
(2)
is zero as soon as \( A \) and \( B \) commute. To this end, prove first that if \( A \) and \( B \) commute, then \( F(t) \) satisfies the differential equation
\[
\dot{F}(t) = (A + B) F(t).
\]
(3)

(e) For the last step, we assume to be known from calculus that given any two \( n \times n \) constant matrices \( A \) and \( B \), the only \( n \times n \) matrix solution \( F(t) \) to the initial value problem
\[
\dot{F}(t) = A F(t) \quad \text{with initial condition} \quad F(0) = B
\]
is
\[
F(t) = e^{A t} B.
\]

Using this fact, prove that if \( A \) and \( B \) commute then equation (1) is true. [Hint: what is \( F(0) \) with \( F(t) \) as defined in equation (2)?]

2. A matrix \( A \) is said to be skew-Hermitian if
\[
A = -A^H.
\]

Show that if \( A \) is skew-Hermitian then the \( n \times n \) matrix
\[
B = A^2 + cI,
\]
where \( c \) is a real scalar and \( I \) is the identity matrix, admits \( n \) orthonormal eigenvectors.
3. How can you find the Schur decomposition of the matrix

\[ A = \begin{bmatrix}
3 & 2 & -4 & 1 \\
0 & 2 & 1 & 2 \\
0 & 0 & -1 & 3 \\
0 & 0 & 0 & 2 \\
\end{bmatrix} \]

and its eigenvalues without using any numerical code?

4. Solve the system of differential equations

\[ \frac{dx}{dt} = \begin{bmatrix}
1 & 1 \\
0 & 2 \\
\end{bmatrix} x \]

with initial conditions \( x(0) = [1 \ 0]^T \). You may leave the answer as a product of matrices, but all the entries of these matrices must be determined.