Exercises

The credit assignment reflects a subjective assessment of difficulty. A typical question can be answered using knowledge of the material combined with some thought and analysis.

1. **Deciding isomorphism** (three credits). What is the computational complexity of recognizing isomorphic abstract simplicial complexes?

2. **Order complex** (two credits). A flag in a simplicial complex $K$ in $\mathbb{R}^d$ is a nested sequence of proper faces, $\sigma_0 < \sigma_1 < \ldots < \sigma_k$. The collection of flags form an abstract simplicial complex $A$ sometimes referred to as the order complex of $K$. Prove that $A$ has a geometric realization in $\mathbb{R}^d$.

3. **Barycentric subdivision** (one credit). Let $K$ consist of a $d$-simplex $\sigma$ and its faces.
   - (i) How many $d$-simplexes belong to the barycentric subdivision, $SdK$?
   - (ii) What is the $d$-dimensional volume of the individual $d$-simplices in $SdK$?

4. **Covering a tree** (one credit). Let $P$ be a finite collection of closed paths that cover a tree, that is, each node and each edge of the tree belongs to at least one path.
   - (i) Prove that the nerve of $P$ is contractible.
   - (ii) Is the nerve still contractible if we allow subtrees in the collection? What about sub-forests?

5. **Nerve of stars** (one credit). Let $K$ be a simplicial complex.
   - (i) Prove that $K$ is a geometric realization of the nerve of the collection of vertex stars in $K$.
   - (ii) Prove that $SdK$ is a geometric realization of the nerve of the collection of stars in $K$.

6. **Helly for boxes** (two credits). The box defined by two points $a = (a_1, a_2, \ldots, a_d)$ and $b = (b_1, b_2, \ldots, b_d)$ in $\mathbb{R}^d$ consists of all points $x$ whose coordinates satisfy $a_i \leq x_i \leq b_i$ for all $i$. Let $F$ be a finite collection of boxes in $\mathbb{R}^d$. Prove that if every pair of boxes has a non-empty intersection then the entire collection has a non-empty intersection.

7. **Alpha complexes** (two credits). Let $S \subseteq \mathbb{R}^d$ be a finite set of points in general position. Recall that Čech($r$) and Alpha($r$) are the Čech
and alpha complexes for radius $r \geq 0$. Is it true that $\text{Alpha}(r) = \text{Čech}(r) \cap \text{Delaunay}$? If yes, prove the following two subcomplex relations. If no, give examples to show which subcomplex relations are not valid.

(i) $\text{Alpha}(r) \subseteq \text{Čech}(r) \cap \text{Delaunay}$.
(ii) $\text{Čech}(r) \cap \text{Delaunay} \subseteq \text{Alpha}(r)$.

8. **Collapsibility** (three credits). Call a simplicial complex *collapsible* if there is a sequence of collapses that reduce the complex to a single vertex. The existence of such a sequence implies that the underlying space of the complex is contractible. Describe a finite 2-dimensional simplicial complex that is not collapsible although its underlying space is contractible.