

## Exercises

The credit assignment reflects a subjective assessment of difficulty. A typical question can be answered using knowledge of the material combined with some thought and analysis.

1. **Hessian** (two credits). Compute the Hessian and, if defined, the index of the origin, which is critical for each function in the list below.
  - (i)  $f(x_1, x_2) = x_1^2 + x_2^2$ .
  - (ii)  $f(x_1, x_2) = x_1x_2$ .
  - (iii)  $f(x_1, x_2) = (x_1 + x_2)^2$ .
  - (iv)  $f(x_1, x_2, x_3) = x_1x_2x_3$ .
  - (v)  $f(x_1, x_2, x_3) = x_1x_2 + x_1x_3 + x_2x_3$ .
  - (vi)  $f(x_1, x_2, x_3) = (x_1 + x_2 + x_3)^2$ .
  
2. **Approximate Morse function** (two credits). Let  $\mathbb{M}$  be a geometrically perfect torus in  $\mathbb{R}^3$ , that is,  $\mathbb{M}$  is swept out by a circle rotating about a line that lies in the same plane but does not intersect the circle. Let  $f : \mathbb{M} \rightarrow \mathbb{R}$  measure height parallel to the symmetry axis and note that  $f$  is not Morse.
  - (i) Describe a Morse function  $g : \mathbb{M} \rightarrow \mathbb{R}$  that differs from  $f$  by an arbitrarily small amount,  $\|f - g\|_\infty < \varepsilon$ .
  - (ii) Draw the Reeb graphs of both functions.
  
3. **Morse-Smale complex** (two credits). Let  $\mathbb{M}$  be the torus in Question 2 and let  $f : \mathbb{M} \rightarrow \mathbb{R}$  measure height along a direction that is almost but not quite parallel to the symmetry axis of the torus.
  - (i) Draw the Morse-Smale complex of the height function.
  - (ii) Give the chain, cycle, boundary groups defined by Floer homology.
  
4. **Quadrangles** (three credits). Let  $\mathbb{M}$  be a 2-manifold and  $f : \mathbb{M} \rightarrow \mathbb{R}$  a Morse-Smale function.
  - (i) Prove that each 2-dimensional cell of the Morse-Smale complex of  $f$  is a quadrangle. In other words, each 2-dimensional cell is an open disk whose boundary can be decomposed into four arcs each glued to an edge in the complex.
  - (ii) Draw a case in which one edge is repeated so that the disk is glued to only three edges but twice to one of the three.

5. **Distance from a point** (three credits). Let  $\mathbb{M}$  be the torus swept out by a unit circle rotating at unit distance from the  $x_3$ -axis. More formally,  $\mathbb{M}$  consists of all solutions to  $x_1^2 + x_2^2 = (2 \pm \sqrt{1 - x_3^2})^2$  in  $\mathbb{R}^3$ . For a point  $z \in \mathbb{R}^3$  consider the function  $f_z : \mathbb{M} \rightarrow \mathbb{R}$  defined by  $f_z(x) = \|x - z\|$ .
- Describe the set of points  $z$  for which  $f_z$  violates property (i) of a Morse function.
  - Describe the set of points  $z$  for which  $f_z$  is not a Morse function.
6. **Non-simple PL critical point** (one credit). Let  $K$  be a triangulation of a 3-manifold and  $f : K \rightarrow \mathbb{R}$  a generic PL function.
- Assuming  $f$  is a PL Morse function, draw the lower links of the four types of simple PL critical points that can occur.
  - Assuming  $f$  is not a PL Morse function, draw the lower link of a non-simple PL critical point.
7. **Lower and upper star filtrations** (one credit). Let  $K$  be a simplicial complex,  $f : K \rightarrow \mathbb{R}$  a generic PL function, and  $f(u_1) < f(u_2) < \dots < f(u_n)$  the ordering of the vertices by function value. For  $0 \leq i \leq n$  let  $K_i$  be the union of lower stars of the first  $i$  vertices and let  $K^i$  be the union of upper stars of the last  $n - i$  vertices. Let  $f(u_i) < t < f(u_{i+1})$ .
- Prove that the sublevel set for threshold  $t$ ,  $f^{-1}(-\infty, t]$ , has the same homotopy type as  $K_i$ .
  - Prove that the superlevel set for threshold  $t$ ,  $f^{-1}[t, \infty)$ , has the same homotopy type as  $K^i$ .
8. **Morse inequalities** (two credits). Recall that the unstable manifolds of a Morse function  $f : \mathbb{M} \rightarrow \mathbb{R}$  are the stable manifolds of  $-f$ . Furthermore, if  $\mathbb{M}$  is a  $d$ -manifold then an index  $p$  critical point of  $f$  is an index  $d - p$  critical point of  $-f$ .
- Use this symmetry to formulate collections of inequalities symmetric to the weak and strong Morse inequalities of  $f$ .
  - Use these inequalities to prove that the Euler characteristic of  $\mathbb{M}$  vanishes if  $d$  is odd.