13.1 Euclidean 1-center

Given a set of points $P = \{p_1, p_2, \ldots, p_n\} \subset \mathbb{R}^d$, the 1-center problem is to determine the smallest radius ball containing $P$. Let $x$ be the center of the ball and let $r$ be its radius. We can express the problem of finding the 1-center as the following optimization problem.

$$\min r, \quad \|x - p_i\| \leq r \quad \forall i \leq n$$

In order to avoid computing square roots, we can reformulate the problem as:

$$\min r^2, \quad \|x - p_i\|^2 \leq r^2 \quad \forall i \leq n$$

We first describe exact solutions to the 1-center problem and then describe approximation algorithms. The running time of all known exact algorithms is exponential in $d$.

13.2 Exact Solutions

In this section, we state three different approaches for finding the minimum enclosing ball.

A parametric search based approach Let $B(x, r)$ be a ball of radius $r$ centered at $x$. The decision version of the 1-center problem is to determine whether, for a given radius $r$,

$$\bigcap_{p_i \in P} B(p_i, r) \neq \emptyset$$

For $d = 2$, this decision problem can be solved in $O(\log n)$ parallel steps using $O(n)$ processors. Thus, using the parametric search technique, we can find the minimum enclosing disc in $O(n \log^2 n)$ time. Unfortunately, this approach does not extend to higher dimensions.
A linear programming based approach  Let us consider the two-dimensional case. Let \( p_i = (a_i, b_i) \), \( x = (x, y) \) and let \( z = r^2 - x^2 - y^2 \). The 1-center problem can be written as

\[
\min z + x^2 + y^2 \\
z \geq -2a_ix - 2b_iy + a_i^2 + b_i^2 \quad \forall i \leq n
\]

By renaming the variables, all constraints have become linear, therefore the feasible region is a convex polyhedron. The objective function is a convex function. Using the prune and search LP paradigm, the 1-center problem can be solved in linear time. This approach extends to higher dimensions, where, for any fixed dimension \( d \), the running time is \( O(d^d n) \).

13.2.1 A randomized algorithm

The smallest enclosing ball of a point set \( P \) in \( d \)-dimensions is defined by \( d + 1 \) or fewer points. We can call this set of \( d + 1 \) or fewer points as the basis of \( P \). Consider a \( d \)-dimensional ball \( B \) passing through a subset of points in \( P \). Clearly, at least one among the points which lie outside \( B \) should be in the basis of \( P \). This observation gives rise to an algorithm which is similar to the randomized incremental algorithm for finding the basis of a Linear program.

\( X \subseteq P \)
\( B(X) : \) Basis of \( X \)
\( Ball(B) : \) Given a set \( B \) of at most \( d + 1 \) points, ball containing \( B \) on its boundary.

\underline{Algorithm 1 MEB(P,G)}

\begin{enumerate}
\item Choose a random Point \( p \in P - G \)
\item \( B = MEB(P - \{p\}, G) \)
\item \textbf{if} \( p \in Ball(B) \) \textbf{then}
\item \quad return \( B \)
\item \textbf{else}
\item \quad return \( MEB(P, Basis(B \cup \{p\})) \)
\item \textbf{end if}
\end{enumerate}

The expected running time of this algorithm is : \( O(ne^{O(\sqrt{(d \log n)})}) \).

13.3 Approximate solutions to the 1-center problem

A 2-Approximation  We present a simple 2-Approximation algorithm for the 1-center problem. The algorithm picks an arbitrary point \( p \in P \) and finds the farthest neighbor of \( p \), say \( p' \in P \). It outputs a ball centered at \( p \) and with radius equal to \( ||p - p'|| \).

Claim 1  Suppose the radius of the minimum enclosing ball is \( r_{opt} \). Algorithm 2 returns a ball of radius \( r \) such that \( r_{opt} \leq r \leq 2 \cdot r_{opt} \)
Algorithm 2 ApproxMEB($P$)
1: Pick an arbitrary point $p \in P$
2: $q = \arg \max_{p' \in P} ||p - p'||$
3: return Ball centered at $p$ of radius $||p - q||$

**Proof:** The ball centered at $p$ and radius $r$ encloses all the points in $P$. This is because $r$ is the distance between $p$ and its farthest neighbor $q$. Thus $r_{opt} \leq r$. Now suppose $c$ is the center of the minimum enclosing ball. Consider the triangle formed by $p, q$ and $c$. From triangular inequality,

$$||p - q|| \leq ||p - c|| + ||c - q|| \leq 2 \cdot r_{opt}$$

A $(1+\epsilon)$ approximation algorithm using $\epsilon$-kernels

**Claim 2** $\text{MEB}(P) = \text{MEB(\text{Conv}(P))}$, where $\text{Conv}(P)$ is the convex hull of $P$

Claim 2 gives us a hint that $\epsilon$-kernels, which approximate the convex-hull, can be used to come up with approximate solutions to the minimum enclosing ball problem. Now, we present an algorithm which essentially constructs the $\epsilon$-kernel of $P$ and uses its minimum enclosing ball to approximate the minimum enclosing ball of $P$. Let $Q \subseteq P$ be an $\epsilon$-kernel of $P$ and let $c_Q$ be the center and $r_Q$ be the radius of $\text{MEB}(Q)$. Clearly, a ball centered at $c_Q$ and radius $(1 + \epsilon) r_Q$ contains the entire point set $P$. Using this observation, here is a $(1 + \epsilon)$-approximation algorithm.

Algorithm 3 ApproxMEB($P$)
1: Construct $Q \subseteq P$, an $\epsilon$-kernel of $P$
2: Construct $\text{MEB}(Q)$
3: Let $c$ be the center of $\text{MEB}(Q)$
4: return Ball with center $c$ and radius $\arg \max_{p \in P} (||p - c||)$

The running time of this algorithm is $O(n \log(1/\epsilon) + 1/\epsilon^{d-1})$ [2].

**Coreset for minimum enclosing ball** We now show that there is a coreset for the minimum enclosing ball of size $\leq 2/\epsilon$.

**Theorem 1** [3] There exists a coreset $S \subseteq P$ of size atmost $2/\epsilon$ such that the radius $r_Q$ of $\text{MEB}(Q)$ is $1/(1 + \epsilon) \cdot r_P$, where $r_P$ is the radius of $\text{MEB}(P)$.

**Proof:** We present the algorithm for constructing the coreset of size $2/\epsilon$ and argue the correctness of the algorithm.

Here is an observation which will be used in proving the correctness of the algorithm.
Algorithm 4 MEB − Coreset(\(P\))

1: Pick an arbitrary point \(p_i \in P\).
2: Let \(S\) be initialized to \(\{p_i\}\).
3: for \(i = 1\) to \(2/\epsilon\) do
4: \(c_i\): Center of MEB(\(S\)).
5: \(p_{i+1}\): \(\arg\max_{q \in P} ||q - c_i||\)
6: \(S = S \cup \{p_{i+1}\}\)
7: end for
8: return \(S\)

Lemma 1 \([1]\) Let \(B\) be the minimum enclosing ball of a point set \(P \subset \mathbb{R}^d\), with center \(c_P\) and radius \(r\), then any closed half-space that contains the center of \(B\), must also contain at least a point from \(P\) that is at a distance \(r\) from the center of \(B\).

Proof: Suppose there exists a closed half-space that contains \(c_P\) and does not contain any point of \(P\) with a distance \(r\) from \(c_P\). Since \(H\) is closed, there exists an \(\delta > 0\), such that the minimum distance between points of \(P \cap H\) and \(H\) is \(> \delta\). Also, fix \(\delta\) such that the distance between any points in \(P \cap H\) from \(c_P\) is atmost \(r - \delta\). This means that we translate \(B\) in the direction perpendicular to \(H\) by \(\delta/2\) (Fig 13.1). After we translate, none of the points of \(P\) will lie on \(B\). This would mean that we can shrink the ball \(B\) contradicting the assumption that \(B\) is a minimum enclosing ball of \(P\). ■

Given Lemma 1, let us prove the correctness of Algorithm 4. Let,

\(R\): Radius of MEB(\(P\)),

\(r_i\): Radius of MEB(\(S_i\)),

\(\lambda_i\): \(r_i/R\),

\(\delta_i\): \(||c_i - c_{i+1}||\),

Figure 13.1: Figure for Lemma 1
Since the radius of the minimum enclosing ball is $R$, there is at least one point $q \in P$ such that $||c_i - q|| \geq R$. Also,

$$r_{i+1} \geq ||c_{i+1} - q|| \geq ||c_i - q|| - ||c_i - c_{i+1}|| \geq R - \delta_i$$

Thus,

$$\lambda_{i+1} R \geq R - \delta_i \quad (13.1)$$

If $\delta_i = 0$, then the maximum distance of any point from $c_i$ is $R$. Hence, we would have found the 1-center. On the other hand, if $\delta_i > 0$, let $H$ be a hyperplane that contains $c_i$ and is orthogonal to $(c_i, c_{i+1})$. Let $H^+$ be the closed halfspace bounded by $H$ and not containing the point $c_{i+1}$. By Lemma 1, there must be a point $p \in S_i \cap H^+$ such that $||c_i - p|| = r_i = \lambda_i R$. Clearly, the triangle formed by $c_i$, $c_{i+1}$ and $p$ is an obtuse-angled triangle, obtuse-angled at $c_i$ (Fig 13.2). Thus $||c_{i+1} - p|| \geq \sqrt{\lambda_i^2 R^2 + \delta_i^2}$.

$$r_{i+1} = \lambda_{i+1} R \geq ||c_{i+1} - p|| \geq \sqrt{\lambda_i^2 R^2 + \delta_i^2}. \quad (13.2)$$

From (13.1) and (13.2) -

$$\lambda_{i+1} R \geq \max \left\{ R - \delta_i, \sqrt{\lambda_i^2 R^2 + \delta_i^2} \right\}$$

$\lambda_{i+1}$ minimizes as a function of $\delta_i$ when

$$R - \delta_i = \sqrt{\lambda_i^2 R^2 + \delta_i^2}$$

$$R^2 - 2\delta_i R + \delta_i^2 = \lambda_i^2 R^2 + \delta_i^2$$

$$\delta_i = \frac{(1 - \lambda_i^2) R}{2}$$

Using (13.1), we get that

$$\lambda_{i+1} \geq \frac{R - \frac{(1 - \lambda_i^2) R}{2}}{R} = \frac{1 + \lambda_i^2}{2}$$
Thus,
\[ \lambda_{i+1} \geq \frac{(1 + \lambda_i^2)}{2} \]
The solution of this recurrence is
\[ \lambda_i \geq 1 - \frac{1}{(1 + i/2)} \]
Thus, if we substitute \( \lambda_i \geq 1 - \epsilon \), we get the inequality \( i \geq \frac{2}{\epsilon} \). This implies that the smallest \( i \) for which \( r_i \geq (1 - \epsilon)R \) is \( \frac{2}{\epsilon} \). Hence proved.

The running time of algorithm 4 is \( O(dn/\epsilon + d^4/\epsilon^2) \) which again is exponential with respect to \( d \). But, in Algorithm 4, if we regard \( S_i \) as a multiset of points, and we choose \( c_i \) to be the centroid of \( S_i \) (taking multiplicity into account), we can find a coreset of size \( 1/\epsilon^2 \) using the same algorithm. Since the algorithm for computing the centroid is polynomial in \( d \), the modified Algorithm 4 would have a running time polynomial in \( d \).

**Claim 3** For any point-set \( P \), we can compute a coreset of size \( 1/\epsilon^2 \) for the minimum enclosing ball problem in \( O(dn/\epsilon + (1/\epsilon^3)) \) time.

### 13.4 1-median problem

In this section, we briefly introduce the 1-median problem and give a sketch of an approximation algorithm. Given a set of points \( P \subset \mathbb{R}^d \), the 1-median problem is to compute a point \( p \in \mathbb{R}^d \) such that the sum of the distances of \( p \) from all points in \( P \) is minimized. That is, compute
\[
\min_{x \in \mathbb{R}^d} \sum_{p_i \in P} ||x - p_i||
\]
The function to be minimized can be viewed as a sum of a set of cones. It is known that a coreset of size \( O(1/\epsilon^{O(d)}) \) can be computed in time \( O(n \log 1/\epsilon + 1/\epsilon^{O(d)}) \).

### References

