1 Overview

In previous lectures we saw a randomized contraction algorithm by Karger for computing (or finding) a global min-cut of a graph. In this lecture we will see a breakthrough result by Nagamochi and Ibaraki for computing the global min-cut. Nagamochi and Ibaraki’s algorithm [2] is deterministic, and runs in \(O(mn)\) time. We will start with a high level description of the algorithm and then see its full details.

2 Problem

Let \(G = (V,E)\) be a graph. For a given pair of vertices \((s,t) \in V\), let \(C(s,t)\) denote the minimum cut separating \(s\) and \(t\). Min-cut of a graph is \(\lambda(G) = \min_{s,t} C(s,t)\). Minimum cut of a graph (also called edge connectivity) can be calculated by running Max-Flow Min-Cut algorithm \(n\) times by taking each vertex as the source. In this lecture, we will see a much faster way to compute min-cut.

2.1 High Level Idea

Nagamochi-Ibaraki algorithm is based on the following beautiful observation. Suppose minimum cut for a vertex pair \((u,v)\) of a graph is the degree cut of either vertex \(v\) or \(u\). By degree-cut we mean the cut formed by taking the edges incident on vertex \(v\). Then, a global min-cut can be found by using the following simple algorithm: Find a pair of vertices \((u,v)\) such that minimum cut separating \((u,v)\) is simply a degree cut of either \(u\) or \(v\) as fast as possible. (Existence of such a pair of vertices was known earlier due to Mader in 1972.) Nagamochi-Ibaraki showed that this can be done in \(O(m)\) time. There are several proofs of this result, and here we will use the proof as given in [1].

2.2 Legal Ordering

For a given subset of vertices \(X\) and \(Y\), \(X \subseteq V\), \(Y \subseteq V\), and \(X \cap Y = \emptyset\), let \(d(X,Y)\) denote the number of edges connecting \(X\) and \(Y\). Let \(d(u) := d(u, V \setminus u)\). For a given graph \(G\), an ordering of vertices \(v_1,v_2,\ldots,v_n\) is said to be a legal ordering if the following condition is satisfied:

\[
d(v_i, \{v_1,v_2,\ldots,v_{i-1}\}) \geq d(v'_i, \{v_1,v_2,\ldots,v_{i-1}\}) \quad \forall v'_i \in V \setminus \{v_1,v_2,\ldots,v_{i-1}\}
\]
In the other words, in a legal ordering the vertex at the position $i$ has the most number of edges to the vertex set in the prefix $1, 2, \ldots, i-1$. By using an appropriate data structure, one can show that legal ordering of a graph can be computed in time $O(m)$, where $m$ is the number of vertices. We will use following simple observations about the legal ordering of a graph $G$.

**Fact 1.** The legal ordering for the graph $G$ remains unchanged if we delete the edge connecting $v_n$ and $v_{n-1}$.

**Fact 2.** The legal ordering for the graph $G \setminus v_n$ is $v_1, v_2, \ldots, v_{n-1}$.

**Fact 3.** The legal ordering for the graph $G \setminus v_{n-1}$ is $v_1, v_2, \ldots, v_{n-2}, v_n$.

Now we will show that given the legal ordering of graph $G$, then the minimum cut between $(v_{n-1}, v_n)$ is equal to the degree cut of $v_n$.

**Lemma 4.** For any graph $G$, and a legal ordering of vertices $v_1, v_2, \ldots, v_n$, $C(v_{n-1}, v_n) = d(v_n)$.

**Proof.** First note that $C(v_{n-1}, v_n) \leq d(v_n)$, since $d(v_n)$ is a valid cut separating $v_n$ and $v_{n-1}$. Hence if we show that $C(v_{n-1}, v_n) \geq d(v_n)$, then the lemma is true. One way of proving the lemma is by contradiction. Assume that $G$ is the minimal counter example graph. A graph is a minimal counter example graph, if there is no other graph $G'$ either with less number of vertices or less number of edges but with same number of vertices as $G$. Clearly, number of vertices in such a counter example graph is at least 3. Furthermore, note that in such a graph $G$ there cannot be an edge connecting $v_{n-1}$ and $v_n$, since removing the edge we get a graph $G'$ for which lemma is true. However, then the lemma would also be true for $G$.

Now consider the graphs $G$ and the graph $G' := G \setminus v_n$ obtained by removing vertex $v_n$. The cardinality of minimum cut separating $v_{n-1}, v_{n-2}$ is greater than the minimum cut separating $v_{n-1}, v_{n-2}$ in $G'$. Therefore,

$$C(v_{n-1}, v_{n-2} : G) \geq C(v_{n-1}, v_{n-2} : G')$$

However, from the observation 2 and the assumption that lemma holds for $G'$ we have,

$$C(v_{n-1}, v_{n-2} : G) \geq C(v_{n-1}, v_{n-2} : G') \geq d(v_{n-1}, V_{n-2})$$

Since there is no edge in between $v_{n-1}$ and $v_{n-1}$ and the legality of the ordering, we can extend the above inequalities as follows:

$$C(v_{n-1}, v_{n-2} : G) \geq C(v_{n-1}, v_{n-2} : G') \geq d(v_{n-1}, V_{n-2}) \geq d(v_n, V_{n-1}) = d(v_n) \quad (2)$$

Next, we consider the graphs $G$ and $G'' := G \setminus v_n$. Since the lemma is also true for $G''$, we get the following inequalities using the same arguments as above.

$$C(v_n, v_{n-2} : G) \geq C(v_n, v_{n-2} : G'') \geq d(v_n, V_{n-2}) \geq d(v_n, V_{n-1}) = d(v_n) \quad (3)$$

Now, observe that

$$C(v_n, v_{n-1}) \geq \min\{C(v_n, v_{n-2}), C(v_{n-1}, v_{n-2})\}$$

The above inequality follows from the simple observation that if we consider the minimum cut separating $v_{n-1}$ and $v_n$, then the vertex $v_{n-2}$ lies either in the component containing $v_n$ or the component containing $v_{n-2}$. Therefore, from equations (2) and (3), we get

$$C(v_n, v_{n-1}) \geq \min\{C(v_n, v_{n-2}), C(v_{n-1}, v_{n-2})\} \geq d(v_n)$$

This contradicts our assumption that $G$ is a minimal counter example graph for the lemma. This completes the proof.
As discussed above, Nagamochi-Ibaraki algorithm uses the legal ordering of a graph to find a min-cut of $G$ as follows. Let $G_i$ be the $i$th iteration of the algorithm. We compute a legal ordering of the vertices of graph $G_i$, and add the degree cut of vertex $v^i_n$ to a list $L$. Here, $v^i_n$ denotes the last vertex which appears in the legal ordering of graph $G_i$. Now, from the above lemma, if the minimum cut of $G_i$ separates $v_{n-1}$ and $v_n$, then degree of cut $v^i_n$ is a minimum cut of $G_i$. Otherwise, minimum cut of $G_i$ remains unchanged by contracting vertices $v^i_{n-1}, v^i_n$. Since global min cut of $G$ should correspond to minimum cut of some $G_i$, by taking the minimum degree cut in the list $L$ we obtain the global min cut of $G$.

### 2.3 Running time

Using an appropriate data structure one can find a legal ordering of $G$ in $O(m)$ time. Since every iteration of Nagamochi-Ibaraki algorithm contracts a pair of vertices there can be at most $n$ iterations in the algorithm. Therefore, the total running time of the algorithm is at most $O(mn)$.

### References
