Discrete Probability

COMPSCI 230 — Discrete Math

April 4, 2017
Outline

1. Probability and Intuition: The Monty Hall Problem
2. Outcomes, Events, and Experiments
3. The Probability Function
The Monty Hall Problem

- Game show with three closed doors 1, 2, 3
- Car behind one door, goats behind the others
- The host knows where the car is
- You get to choose one door, and you get the prize behind it
- Assume that you want the car
- Say you pick door 1
The Monty Hall Problem

- You picked door 1
- Instead of opening door 1 the host opens door 3, which has a goat
  [The host never opens the door with the car]
- Host asks, ”Do you want to pick door 2 instead of 1?”
- **Should you switch to door 2?**
Monty Hall Problem

Is it best to switch to door 2?  (Not Graded)

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<tbody>
<tr>
<td>A</td>
<td>Yes</td>
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<tr>
<td>B</td>
<td>No</td>
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<tr>
<td>C</td>
<td>Makes no difference</td>
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Sample Space

What is a sample space? (Graded)

A: A set of outcomes
B: A space of events
C: A sequence of outcomes
D: A set of events
E: None of the above
Probability Function

What is the domain of a probability function $P(\cdot)$?

Pick one (Graded)

A: $[0, 1]
B: A sample space
C: An experiment
D: A space of events
E: $\mathbb{R}$
Equally Likely Outcomes

What is the probability of event $A$ in sample space $S$ if all outcomes are equally likely?

Pick one

A: $|S|^A$
B: $|A|^S$
C: $|S|/|A|$
D: $|A|/|S|$
E: $|S| - |A|$
Intuition?

• Intuition seems to suggest that at the time you are given the choice between door 1 and door 2 there is no reason to prefer one over the other
• The car could be behind either door, you have no other information, right?
• So switching seems to make no difference
• *If I am given a choice between two closed doors, I have no reason to prefer one over the other*
• Is that really true?
• If not, what is wrong with this intuition?
The Monty Hall 98-Door Problem

- Game show with 98 closed doors 1, ..., 98
- Car behind one door, goats behind the others
- The host knows where the car is
- You get to choose one door, and you get the prize behind it
- Say you pick door 1
The Monty Hall 98-Door Problem

- You picked door 1
- Instead of opening door 1 the host opens 96 other doors, all of which have goats, and leaves door 64 shut
- Host asks, "Do you want to pick door 64 instead of 1?"
- **Should you switch to door 64?**
98-Door Monty Hall Problem

Is it best to switch to door 64? (Not Graded)

A: Yes
B: No
C: Makes no difference
Revisiting Intuition

• With 98 doors, it is unlikely that door 1 is the correct choice
• *It is more likely that the car is behind one of the other doors*
• 97 times more likely!
• The host knows where the car is
• He eliminates all but one of the other doors for you
• *It is still more likely that the car is behind door 64*
• *97 times more likely!*
• Yes, you should switch
Back to 3 Doors

• With 3 doors, it is somewhat unlikely that door 1 is the correct choice (one chance out of three)
• *It is more likely that the car is behind one of the other doors*
• Twice as likely!
• The host knows where the car is
• He eliminates all but one of the other doors for you
• It is still more likely that the car is behind door 2
• **Twice as likely!**
Monty Hall Probability

What is the probability that the car is behind door 2 after the host has opened door 3 with the goat?

Pick one (Not Graded)

A: 1/3
B: 1/2
C: 2/3
D: 5/6
E: 3/4
Monty Hall Problem

Is it best to switch to door 2?  (Not Graded)

A: Yes
B: No
C: Makes no difference
3-Door Case Conclusion

- You double your chances by switching to door 2
- The car could still be behind door 1
- ... but is half as likely to be there than to be behind door 2
- This means that if you play the same game 100 times, you win approximately twice as often if you always take the offer to switch doors than if you do not
- You win about a third of the time if you never switch
- You win about two thirds of the time if you always switch
- This is called the frequentist interpretation of probability
- Other interpretations are possible
What Can Go Wrong with Intuition?

- We may not notice that the host’s knowledge injects asymmetry into the choice between the two remaining close doors.
- With three doors the asymmetry is relatively small... ...but far from trivial: It is twice more likely that the car is behind door 2 than behind door 1.
- With 98 doors the asymmetry is more visible: it is 97 times more likely that the car is behind door 64 than behind door 1.
- The host injects so much information into the scenario that we cannot fail to notice.

**Intuition: Use with caution**

- The calculus of probability formalizes reasoning with uncertain information and makes it quantitative.
- We no longer need to rely on intuition only.
Discrete Probability

• The calculus of probability helps reason about *odds* formally, just as logic helps reason about *facts* formally.
• If we know probability we can make better decisions under uncertainty.
• “Discrete:” countably many possible outcomes.
  Often but not always finitely many.
• Continuous probability: same conceptual foundation but technically trickier.
• Most continuous outcomes have probability 0.
• Only discrete probability in COMPSCI 230.
Outcomes and Events

- **Sample Space**: a finite or countable set $\mathcal{S}$
  Example: $\{1, 2, 3, 4, 5, 6\}$ (but need not be numbers)

- **Outcome**: an item from a sample space, $O \in \mathcal{S}$
  Example: 5

- **Event**: a subset of the sample space, $A \subseteq \mathcal{S}$
  Examples: $\{2, 5\}$, $\emptyset$, $\mathcal{S}$, $\{5\}$
  The last one is called a *singleton event*

- **Event Space**: the set of all events, $\mathcal{E} = \mathcal{P}(\mathcal{S})$
  The event space is the power set of the sample space
  In the example above, $|\mathcal{E}| = 64$

- **Experiment**: a procedure that produces outcomes (not events!)
  Example: Single roll of one die
Events and Experiments

\( S = \{1, 2, 3, 4, 5, 6\} \)
\( E = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{1, 6\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{2, 6\}, \{3, 4\}, \{3, 5\}, \{3, 6\}, \{4, 5\}, \{4, 6\}, \{5, 6\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \{1, 2, 6\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 3, 6\}, \{1, 4, 5\}, \{1, 4, 6\}, \{1, 5, 6\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 3, 6\}, \{2, 4, 5\}, \{2, 4, 6\}, \{2, 5, 6\}, \{3, 4, 5\}, \{3, 4, 6\}, \{3, 5, 6\}, \{4, 5, 6\}, \{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{1, 2, 3, 6\}, \{1, 2, 4, 5\}, \{1, 2, 4, 6\}, \{1, 2, 5, 6\}, \{1, 3, 4, 5\}, \{1, 3, 4, 6\}, \{1, 3, 5, 6\}, \{1, 4, 5, 6\}, \{2, 3, 4, 5\}, \{2, 3, 4, 6\}, \{2, 3, 5, 6\}, \{2, 4, 5, 6\}, \{3, 4, 5, 6\}, \{1, 2, 3, 4, 5\}, \{1, 2, 3, 4, 6\}, \{1, 2, 3, 5, 6\}, \{1, 2, 4, 5, 6\}, \{1, 3, 4, 5, 6\}, \{2, 3, 4, 5, 6\}, \{1, 2, 3, 4, 5, 6\}\)

- \({6 \choose k}\) events with \(k\) outcomes
- Event \(\{3, 4\}\) is a set and represents the following alternative:
  
  *A single roll of one die produces either outcome 3 or outcome 4*

- An event is a *disjunction* (“or”) of outcomes
- Nothing to do with multiple outcomes or repeated experiments
- One experiment always produces one outcome
- When outcome \(x\) occurs, so do all events that contain \(x\)
- Two events with empty intersection are *mutually exclusive*
Outcomes and Events

How many events for the experiment of rolling a single die once occur when the outcome is 3?

Pick one

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<td>C</td>
<td>6</td>
</tr>
<tr>
<td>D</td>
<td>16</td>
</tr>
<tr>
<td>E</td>
<td>32</td>
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(Not Graded)
Outcomes, Events, and Experiments

### Compound Experiments

- Roll of one die has sample space \( \mathcal{S}_D = \{1, 2, 3, 4, 5, 6\} \) and event space \( \mathcal{E}_D = \mathcal{P}(\mathcal{S}_D) \) (so that \( |\mathcal{S}_D| = 6 \) and \( |\mathcal{E}_D| = 64 \))
- Flip of one coin has sample space \( \mathcal{S}_C = \{H, T\} \) and event space \( \mathcal{E}_C = \mathcal{P}(\mathcal{S}_C) \) (so that \( |\mathcal{S}_E| = 2 \) and \( |\mathcal{E}_C| = 4 \))
- Rolling a die and flipping a coin is a **compound experiment**
- It has sample space \( \mathcal{S} = \mathcal{S}_D \times \mathcal{S}_C \) and event space \( \mathcal{E} = \mathcal{P}(\mathcal{S}) \) (so that \( |\mathcal{S}| = 12 \) and \( |\mathcal{E}| = 4096 \))
- No need to say if roll or flip comes first in time (they could be simultaneous as well)
- However, the outcomes are listed as ordered pairs (or sequences if there are more experiments)
- Example outcomes: \((4, H), (1, H), (2, T)\)
- Example events: \(\emptyset, \{(4, H), (3, H), (5, T), (6, T)\}, \{(3, T)\}\)

The last one is a singleton event
Repeated Experiments

• A repeated experiment is a compound experiment where all the individual sample spaces and probability functions are equal
• Example: Rolling a die three times: $\mathcal{S} = \mathcal{S}_D \times \mathcal{S}_D \times \mathcal{S}_D$
• Indistinguishable from rolling three identical dice
Event Spaces

Let $|\mathcal{S}_D| = 6$ and let $\mathcal{E}$ be the event space of the repeated experiment with sample space $\mathcal{S}_D \times \mathcal{S}_D \times \mathcal{S}_D$ (rolling three dice).
Assume that the algebra below is correct.

What is $|\mathcal{E}|$?

(Not Graded)

A: $2 \times 6 \times 3 = 36$
B: $(2^6)^3 = 64^3 = 262,144$
C: $(6^2)^3 = 36^3 = 46,656$
D: $6^{2^3} = 6^8 = 1,679,616$
E: $2^{6^3} = 2^{216} \approx 10^{56}$
Probability Function

- We are eventually interested in probabilities of events.
- We start by assigning probabilities to outcomes.
- Also known as a (discrete) probability distribution.
- A function $P : \mathbb{S} \rightarrow \mathbb{R}$ that satisfies the axioms of probability:

$$\forall O \in \mathbb{S} : P(O) \geq 0$$
$$\sum_{O \in \mathbb{S}} P(O) = 1$$

- Probability of events (as opposed to outcomes):
  - $\forall A \in \mathbb{E} : \text{prob}(A) = \sum_{O \in A} P(O)$
  - The domain of $P$ is $\mathbb{S}$. The domain of prob is $\mathbb{E}$
  - Connection: $\text{prob}(\{O\}) = P(O)$ for all $O \in \mathbb{S}$
  - [Note that $A \in \mathbb{E}$ but $A \subseteq \mathbb{S}$]
Unbiased Coin

\[ S = \{H, T\} \]
\[ \mathcal{E} = \{\emptyset, \{H\}, \{T\}, \{H, T\}\} \]
\[ |S| = 2 \Rightarrow |\mathcal{E}| = 2^2 = 4 \]

\[ P(H) = P(T) = 1/2 \]

• Biased coin: Same, but

\[ P(H) = p, \quad P(T) = q, \quad p + q = 1 \quad p, q \geq 0 \]
Two Biased Coins

\begin{align*}
\mathcal{S} &= \{(H, H), (H, T), (T, H), (T, T)\} \\
\mathcal{E} &= \{\emptyset, \\
&\quad \{(H, H)\}, \{(H, T)\}, \{(T, H)\}, \{(T, T)\}, (\leftarrow \text{singletons!}) \\
&\quad \{(H, H), (H, T)\}, \{(H, H), (T, H)\}, \{(H, H), (T, T)\}, \\
&\quad \{(H, T), (T, H)\}, \{(H, T), (T, T)\}, \{(T, H), (T, T)\}, \\
&\quad \{(H, H), (H, T), (T, H)\}, \{(H, H), (H, T), (T, T)\}, \\
&\quad \{(H, H), (T, H), (T, T)\}, \{(H, T), (T, H), (T, T)\}, \\
&\quad \{(H, H), (H, T), (T, H), (T, T)\}\}
\end{align*}

\begin{align*}
|\mathcal{S}| = 4 &\Rightarrow |\mathcal{E}| = 2^4 = 16 \\
\binom{4}{k} \text{ events with } k \text{ outcomes}
\end{align*}

Possible (but not only) probability assignment ("independent outcomes"):

\begin{align*}
P((H, H)) &= p_1 p_2 \\
P((H, T)) &= p_1 q_2 \\
P((T, H)) &= q_1 p_2 \\
P((T, T)) &= q_1 q_2
\end{align*}

\begin{align*}
p_1 + q_1 &= p_2 + q_2 = 1 \Rightarrow p_1 p_2 + p_1 q_2 + q_1 p_2 + q_1 q_2 = \\
p_1(p_2 + q_2) + q_1(p_2 + q_2) &= (p_1 + q_1)(p_2 + q_2) = 1 \cdot 1 = 1
\end{align*}