

3 Equivalence Relations

In this section, we generalize the problem of counting subsets in two different ways. We also introduce equivalence relations as a new concept.

Labeling. To begin, we ask how many ways are there to label three of five elements red and the remaining two elements blue? Without loss of generality, we can call our elements A, B, C, D, E . A *labeling* is a function that associates a color to each element. Suppose we look at a permutation of the five elements and agree to color the first three red and the last two blue. Then the permutation $ABDCE$ would correspond to coloring A, B, D red and C, E blue. However, we get the same labeling with other permutations, namely with all permutations that begin with A, B, D , in any order, and end with C, E , also in any order:

$$\begin{array}{lll} ABD; CE & BAD; CE & DAB; CE \\ ABD; EC & BAD; EC & DAB; EC \\ ADB; CE & BDA; CE & DBA; CE \\ ADB; EC & BDA; EC & DBA; EC . \end{array}$$

Indeed, we have $3!2! = 12$ permutations that give the same labeling, simply because there are $3!$ ways to order the red elements and $2!$ ways to order the blue elements. Similarly, every other labeling corresponds to 12 permutations. In total, we have $5! = 120$ permutations of five elements. The set of 120 permutations can thus be partitioned into $\frac{120}{12} = 10$ blocks such that any two permutations in the same block give the same labeling. Any two permutations from different blocks give different labelings, which implies that the number of different labelings is 10. More generally, the number of ways we can label k of n elements red and the remaining $n - k$ elements blue is $\frac{n!}{k!(n-k)!} = \binom{n}{k}$. This is also the number of k -element subsets of a set of n elements.

Now suppose we have three labels, red, green, and blue. We count the number of different labelings by dividing the total number of orderings by the orderings within in the color classes. There are $n!$ permutations of the n elements. We want i elements red, j elements blue, and $\ell = n - i - j$ elements green. We agree that a permutation corresponds to the labeling we get by coloring the first i elements red, the next j elements blue, and the last ℓ elements green. The number of repeated labelings is thus $i!$ times $j!$ times $\ell!$ and we have $\frac{n!}{i!j!\ell!}$ different labelings. This generalizes in the obvious way to k colors.

Equivalence relations. We now formalize the above method of counting. A *relation* on a set S is a collection R of ordered pairs, $(x, y) \in S \times S$. We write $x \sim y$ if the pair (x, y) is in R . Note the similarity between a function and a relation. More specifically, a relation is like a function from S to S , except that it can assign any number of images to any one element, not just one like it is required for a function. The relation is

- *reflexive* if $x \sim x$ for all $x \in S$;
- *symmetric* if $x \sim y$ implies $y \sim x$;
- *transitive* if $x \sim y$ and $y \sim z$ imply $x \sim z$.

We say that the relation is an *equivalence relation* if R is reflexive, symmetric, and transitive. Not all relations are equivalence relations. Indeed, there are relations that have none of the above three properties. If S is a set and R an equivalence relation on S , then the *equivalence class* of an element $x \in S$ is

$$[x] = \{y \in S \mid y \sim x\}.$$

We note here that if $x \sim y$ then $[x] = [y]$. This implies that any two equivalence classes are either the same or they are disjoint. Indeed, if $[x] \neq [z]$ and y belongs to both then $[x] = [y] = [z]$, which contradicts the assumption.

In the above labeling example, S is the set of permutations of the elements A, B, C, D, E and two permutations are equivalent if they give the same labeling. Verify that this indeed defines an equivalence relation. Recalling that we color the first three elements red and the last two blue, the equivalence classes are $[ABC; DE]$, $[ABD; CE]$, $[ABE; CD]$, $[ACD; BE]$, $[ACE; BD]$, $[ADE; BC]$, $[BCD; AE]$, $[BCE; AD]$, $[BDE; AC]$, $[CDE; AB]$.

An example: modular arithmetic. We say an integer a is *congruent* to another integer b modulo a positive integer n , denoted as $a \equiv b \pmod{n}$, if $b - a$ is an integer multiple of n . To illustrate this definition, let $n = 3$ and let S be the set of integers from 0 to 11. Then $x \equiv y \pmod{3}$ if x and y both belong to $S_0 = \{0, 3, 6, 9\}$ or both belong to $S_1 = \{1, 4, 7, 10\}$ or both belong to $S_2 = \{2, 5, 8, 11\}$. This can be easily verified by testing each pair. Congruence modulo 3 is in fact an equivalence relation on S . To see this, we show that congruence modulo 3 satisfies the three required properties.

reflexive. Since $x - x = 0 \cdot 3$, we know that $x \equiv x \pmod{3}$.
symmetric. If $x \equiv y \pmod{3}$ then $y - x = 3k$ for some integer k . Hence, $x - y = -3k$, and since $-k$ is an integer, we have $y \equiv x \pmod{3}$.

transitive. Let $x = y \pmod 3$ and $y = z \pmod 3$. Then there are integers k and ℓ such that $y - x = 3k$ and $z - y = 3\ell$. It follows that $z - x = 3k + 3\ell = 3(k + \ell)$, and since $k + \ell$ is an integer, we have $x = z \pmod 3$.

More generally, congruence modulo n is an equivalence relation on the integers.

Block decomposition. An equivalence class of elements is sometimes called a *block*. The importance of equivalence relations is based on the fact that the blocks partition the set. Recall that this means that the union of blocks is S and any two blocks have an empty intersection. But we already learned that different blocks are disjoint.

BLOCK PARTITION THEOREM. Let R be an equivalence relation on a set S . Then the blocks partition S .

PROOF. In order to prove that $\bigcup_{x \in S} [x] = S$, we need to show two things, namely $\bigcup_{x \in S} [x] \subseteq S$ and $\bigcup_{x \in S} [x] \supseteq S$. Each $[x]$ is a subset of S which implies the first inclusion. Furthermore, each $x \in S$ belongs to $[x]$ which implies the second inclusion. \square

Symmetrically, a partition of S defines an equivalence relation. If the blocks are all of the same size then it is easy to count them.

QUOTIENT PRINCIPLE. If a set S of size p can be partitioned into q blocks of size r each, then $p = qr$ or, equivalently, $q = \frac{p}{r}$.

Multisets. The difference between a set and a *multiset* is that the latter may contain the same element multiple times. In other words, a multiset is an unordered collection of elements, possibly with repetitions. We can list the repetitions,

$$\langle\langle c, o, l, o, r \rangle\rangle$$

or we can specify the multiplicities,

$$m(c) = 1, m(o) = 2, m(l) = 1, m(r) = 1.$$

The *size* of a multiset is the sum of the multiplicities. We show how to count multisets by considering an example: the ways to distribute k (identical) books among n (different) shelves. The number of ways is equal to

- the number of size- k multisets of the n shelves;
- the number of ways to write k as a sum of n non-negative integers.

We count the ways to write k as a sum of n non-negative integers as follows. Choose the first integer of the sum to be i . Now we have reduced the problem to counting the ways to write $k - i$ as the sum of $n - 1$ non-negative integers. For small values of n , we can do this. For example, let $n = 3$. Then, we have $i + j + \ell = k$. The choices for i are from 0 to k . Once i is chosen, the choices for j are fewer, namely from 0 to $k - i$. Finally, if i and j are chosen then ℓ is determined, namely $\ell = k - i - j$. The number of ways to write k as a sum of three non-negative integers is therefore

$$\begin{aligned} \sum_{i=0}^k \sum_{j=0}^{k-i} 1 &= \sum_{i=0}^k (k - i + 1) \\ &= \sum_{i=1}^{k+1} i \\ &= \binom{k+2}{2}. \end{aligned}$$

To get the formula for n equal to 4 or larger, we form sums of higher-degree binomial coefficients. However, there is a simpler way to find the solution. Suppose we line up our n books, then place $k - 1$ dividers between them. The number of books between the i -th and the $(i - 1)$ -st dividers is equal to the number of books on the i -th shelf; see Figure 5. We thus have $n + k - 1$ objects, k books plus $n - 1$ dividers. The number of ways to choose $n - 1$



Figure 5: Five books on four shelves: two books on the first shelf, one on the second, an empty third shelf, and two books on the fourth and last shelf. This figure represents $5 = 2 + 1 + 0 + 2$.

dividers from $n + k - 1$ objects is $\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$. We can easily see that this formula agrees with the result we found for $n = 3$.

Summary. We defined relations and equivalence relations, investigating several examples of both. In particular, modular arithmetic creates equivalence classes of the integers. Finally, we looked at multisets, and saw that counting the number of size- k multisets of n elements is equal to the number of ways to write k as a sum of n non-negative integers.