Part I. LS approximation (I): special case

1. Continuous LS approximation with trigonometric polynomials.
   Let \( f(x) = \alpha x^2 + \beta x \), with \( \alpha, \beta \) as real scalars.
   (a) Formulate in linear combination form the LS approximation of \( f(x) \) in \( T_n[0, 2\pi] \) for \( n > 0 \).
   (b) Formulate a system of linear equations for the LS solution and give the solution.
   (c) Extend the results to arbitrary finite interval \([a, b]\).

2. Conversion to the exponential format. Let \( V \) be the vector space of continuous complex functions
   \[ V_c[a, b] = \{ c(x) + i \cdot s(x) \mid c, s \in C[a, b] \} \]
   over the complex field.
   (a) Let
   \[ \langle f, g \rangle = \int_a^b \bar{f}(x)g(x)dx, \quad f, g \in V_c[a, b]. \]
   Then it is an inner product on \( V_c[a, b] \). Verify that the set of functions
   \[ \{ e^{-ikx} \mid x \in [a, b], k \in Z \} \subset V_c[a, b] \]
   are orthogonal on the interval \([a, b] = [0, 2\pi] + \mu\), where \( \mu \) is an arbitrary translation. Find the normalization constants.
   (b) Let \([a, b] = [0, 2\pi]\). Let \( EXP_n = \text{span}(\{ e^{-ikx} \mid -n \leq k \leq n \}) \subset V_c \). Find the representation of function \( f \in T_n \subset EXP_n \) as a linear combination of \( e^{ikx} \).

3. Discrete LS approximation with trigonometric polynomials
   Let the data be \((x_j, y_j)\), with \( x_j \) equally spaced in \([0, 2\pi]\),
   \[ x_j = j(2\pi)/(2n + 1), \quad j = 0 : 2n. \]
(a) Formulate in matrix form
\[ \min_{a,b} \| Aa + Bb - y \|_2 \]
the problem of the LS curve fitting on the data with trigonometric polynomials in \( T_m[0, 2\pi] \), where \( m \) is not necessarily equal to \( n \).

(b) Formulate a system of linear equations for the LS solution.

(c) Let \( V_d \) be the vector space of discrete complex functions defined as follows
\[ V_d = \{ c(x_j) + i \cdot s(x_j), j = 0 : 2n \mid c(x_j), s(x_j) \in \mathbb{R} \} \]
over the complex field. Let
\[ \langle u, v \rangle = \sum_{j=0}^{2n} \bar{u}(x_i)v(x_i) = u^Hv, \quad u, v \in V_d, \]
be an inner product on \( V_d \). Verify that the set of discrete functions
\[ \{ e^{-ikx_j}, k = -m : m \mid j = 0 : 2n \} \subset V_d \]
form an orthogonal basis of \( V_d \). Find the normalization constants. (Hint: use the formulae for geometric sum).

(d) Deduce that the real matrix \((A, B)\) has orthogonal columns when \( m \leq n \). Describe a simple procedure to compute the coefficient vectors \( a \) and \( b \), and provide the computation complexity of arithmetic operations in terms of \( n \) and \( m \leq n \).

(e) Find a connection between the matrix \((A, B)\) and the DFT matrix, and describe a fast approach to determining the LS fitting curve in \( T_m \), with \( n/2 \leq m \leq n \).

4. Find the differences between continuous LS approximation and discrete LS approximation with trigonometric polynomials. In particular, consider the case that \( y = \alpha x^2 + \beta x \) with \( x \in [0, 2\pi] \) in the continuous case and \((x_i = i(2\pi)/(2n + 1), y_i)\) in the discrete case.
Part II. Review: Numerical Solution of Linear Equations

\[ Ax = b, \]

where \( A \) is an \( m \times n \) real or complex matrix.

1. Give an example so that the solution does not exist.

2. Give an example so that the solution exists but not unique.

3. Columnwise elimination with lower triangular matrix. Let \( A \) be a real \( m \times 1 \) nonzero matrix. Let \( b = e_1 \), where \( e_1 = [1, 0, \cdots, 0]^T \). Determine a permutation matrix \( P_r \), a lower unit triangular matrix \( L \) (which is a lower triangular matrix with all diagonal elements equal to 1), and a scalar \( u \) so that

\[ L^{-1}(P_rA) = u \cdot e_1. \]

4. Rank-revealing LU factorization. For any \( A \), there exist permutation matrices \( P_r \) and \( P_c \), a lower unit triangular matrix \( L \) and an \( m \times n \) upper triangular matrix \( U \) such that

\[ P_rAP_c = LU, \]

where

\[ U = \begin{bmatrix} U_{11} & U_{12} \\ 0 & 0 \end{bmatrix} \]

and \( U_{11} \) is non-singular. The size of \( U_{11} \) is the rank of \( A \). Let \( y \) be the solution of equations

\[ Ly = P_r b. \]

How can we tell from \( y \) and \( U \) when the solution \( x \) exists and when the solution exists but not unique.

5. Columnwise elimination with Householder matrix. Let \( A \) be a real \( m \times 1 \) nonzero matrix. Let \( b = e_1 \). Determine the Householder matrix of the form

\[ H = I - 2uu^T, \quad u^Tu = 1 \]

which is symmetric and orthogonal, and a scalar \( r \) such that

\[ HA = r \cdot e_1. \]
6. **Rank-revealing QR factorization.** For any $A$, there exists a permutation matrix $P$, a unitary matrix $Q$ and an $m \times n$ upper triangular matrix $R$ such that

$$AP = QR,$$

where

$$R = \begin{bmatrix} R_{11} & R_{12} \\ 0 & 0 \end{bmatrix},$$

and $R_{11}$ is non-singular. How can we tell from $y = Q^H$ and $R$ when the solution $x$ exists and when the solution exists but not unique.

7. **Circulant deconvolution.** Let $A$ be a circulant matrix of order $n$. Describe a fast procedure for determining the existence and uniqueness of solution. When the solution exists, find out the solution $x$ with minimum 2-norm.

**Part III. Numerical Solution of Discrete LS Approximation**

$$\min_x \|Ax - b\|_2$$

where

$$A = [f_j(t_i)]_{i=1:n,j=1:m}, \quad b = [y_i]_{i=1:n}$$

in curve fitting with basis functions $f_j, j = 1 : m$, on data $(t_i, y_i), i = 1 : n$, and $x$ are the coefficients to be determined.

1. Verify that there always exists LS solution.
2. Give a condition for the uniqueness.
3. Describe a procedure for LS solution via $QR$ factorization of $A$. 

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