Algorithm Prototype and Numerical Experiments in MATLAB

1. Implement the iterative method you designed for computation of the square root, to single precision.
   Requirement:
   (a) the seed table for initial values is of size no bigger than 16.
   (b) report the iteration number needed for all mantissa numbers in $[1/2, 1)$ and the number of operations for each iteration.
   (c) generate 10,000 nonnegative numbers randomly, compute their square roots, plot the residual distribution.

2. Numerical computation of definite integral
   $$\int_{a}^{b} f(x)dx,$$
   for any $f \in C^4[a, b]$. The input includes
   - the interface with a function evaluation subroutine,
   - an initial sampling distribution (partition)
   - a numerical error tolerance,
   - a maximum number of sample points.
   The output includes
   - a numerical value of the integration,
   - an estimation of the error,
   - the sample distribution generated,
   - the number of function evaluations invoked,
   - error flag.
   Requirement:
   (a) sampling is adaptive to error tolerance (based on automatic error estimate) and adaptive to the change rate of the integrand.
   (b) visualize the sampling distribution in response to the change rate of the integrand.
   Cf. the initial draft AdaptiveQuadrature.m
3. Variational method for the PDE BVP

\[
\frac{\partial}{\partial x} (p(x,y)u_x(x,y)) + \frac{\partial}{\partial y} (q(x,y)u_y(x,y)) - r(x,y)u(x,y) = -f(x,y),
\]

\[p, q, -r \geq 0, \quad x \in D,\]

\[u(x,y) = 0, \quad (x,y) \in \partial D,\]

The input includes

- an interface to function evaluation subroutines for \(p, q, r, f\) and the basis functions \(\phi_k\) for the subspace \(V_N\).
- an interface to a basis function evaluation subroutine.

The output includes

(a) 2-D visualization of the matrix for the normal equation.
(b) 3-D visualization of the numerical approximation to the PDE solution.

To test the implementation, use the specific case in HW-6 with \(N = n^2\), \(n = 35, 70\), and use the three specific sets of \(\{\phi_k | k = 1 : 100\}\) considered in HW-6.

4. Numerical solution of the heat equation

\[
PDE \quad \frac{\partial u}{\partial t} = \nabla^2 u, \quad x \in (0,1), \quad t \geq 0
\]

\[
IC \quad u(x,0) = g(x), \quad x \in [0,1], \quad g(0) = g(1) = 0
\]

\[
BC \quad u(a,t) = u(b,t) = 0, \quad t > 0.
\]

Requirement: use the trapezoidal rule in time integration to convert the problem into a sequence of ODE BVPs.

For the input, cf. the last problem; for the output, provide 3-D visualization of the numerical approximation to the heat equation solution.
Image Restoration

The following matrix equation

\[ G = C_x H C_y, \quad F, H \in \mathbb{R}^{m \times n}, \]

is a discretized version from an model in integral form for the observed blurred image \( G \), the blurring operators \( C_x \) and \( C_y \) in \( x \) and \( y \) directions, respectively, and the source image \( H \) to be restored. In an ideal case, the blurring operators are translation invariant. In other words, \( C_x \) and \( C_y \) are Toeplitz matrices. If the boundary conditions are periodic, \( C_x \) and \( C_y \) are circulant.

a. Noiseless restoration with numerically nonsingular blurring operators.
   Experiment with given circulant operators and an image in ResImage1.mat.

b. Noiseless restoration with numerically singular blurring operators.
   The problem is posed as a LS problem instead
   \[ \min_H \| \tilde{G} - C_x H C_y \|_F \]
   where the matrix norm \( \| \cdot \|_F \) is defined as follows
   \[ \| A \|_F^2 = \sum_j \| Ae_j \|_2^2. \]
   Experiment with given circulant operators and an image in ResImage2.mat.

c. Noisy restoration with nearly singular operators.
   In this case, there are errors/perturbations/noise in the observed image from different sources,
   \[ \tilde{G} = G + E. \]
   Consider the simplest case that the operators are circulant, Carry out numerical experiments, using the given operators and an image in ResImage3.mat,
   \( \diamond \) without regulating the weights on the operator eigenvalues.
   \( \diamond \) with regulation on the eigenvalues.
   Give some explanations to what you have observed.