CS 296.4 Sample Midterm Exam

This exam is closed book, closed notes.
There are 5 problems. Points add up to 100.
WRITE YOUR NAME ON THIS PAGE NOW.
Be telegraphic in your answers.

October 29, 2002

1. (20 points) If the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are orthonormal, and if you know that for a given vector $\mathbf{a}$ there exist coefficients $\alpha_1, \ldots, \alpha_n$ such that

$$\mathbf{a} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n \ ,$$

then the coefficients α_j can be computed as

$$\alpha_j = \mathbf{v}_j^T \mathbf{a} \ .$$

(a) (10 points) How can you compute the coefficients α_j if the vectors $\mathbf{v}_j$ are linearly independent but not orthogonal?

(b) (10 points) If the vectors $\mathbf{v}_j$ are linearly independent but not orthogonal, are the coefficients α_j unique? Why or why not?

2. (20 points) Let A be any $m \times n$ matrix. No assumption is made about the relative size of m and n.

(a) (10 points) Show that the matrix $AA^\dagger$, where $\dagger$ denotes the pseudoinverse, is a projection matrix, that is, show that it projects vectors onto some subspace.

(b) (5 points) What subspace, related to the matrix A, does $AA^\dagger$ project vectors onto?

(c) (5 points) By switching the roles of A and $A^\dagger$, your argument above shows that also $A^\dagger A$ is a projection matrix. What subspace, related to the matrix A, does $A^\dagger A$ project vectors onto?

3. (20 points) Let N be the null space of an $m \times n$ matrix A with $m > n$, and let R be its range.

(a) (3 points) What is the rowspace W of A^T?

(b) (3 points) What is the left null space L of A^T ?

(c) (8 points) Prove that the range of A is the same as the range of AA^T.

(d) (6 points) Prove that AA^T is not invertible.

4. (20 points) The set of all 2×2 matrices forms a linear space of dimension four. A basis for this space is for instance

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} .$$

A 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is said to be *constant-sum* if the row sums and column sums are all equal to each other:

$$a_{11} + a_{12} = a_{21} + a_{22} = a_{11} + a_{21} = a_{12} + a_{22} .$$

(a) (10 points) What is the dimension of the space of all 2×2 constant-sum matrices, and why?

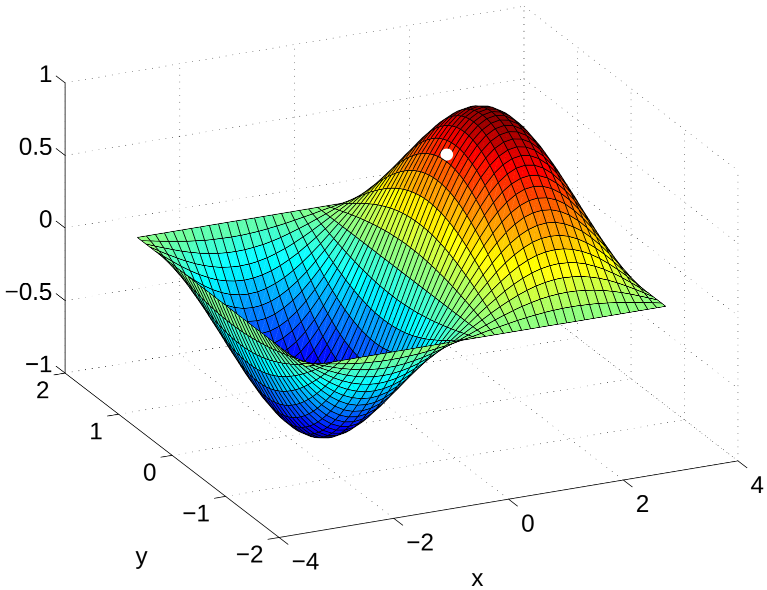
(b) (10 points) Find a basis for the space of all 2×2 constant-sum matrices.

5. (20 points)

The figure shows a plot of the function

$$f(x, y) = \sin x \cos y$$

for $-\pi \le x \le \pi$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. The white dot is the point on the graph of f at $\mathbf{x}_0 = (x_0, y_0)^T = (\frac{\pi}{4}, 0)^T$, where f has value $f(\frac{\pi}{4}, 0) = \frac{\sqrt{2}}{2}$.



Briefly explain your answers below.

(a) (6 points) Where will Newton's method be after one step, when applied to the function $f(x, y)$ with initial point $\mathbf{x}_0$?

(b) (4 points) What point will Newton's method converge to, when applied to the function $f(x, y)$ with initial point $\mathbf{x}_0$?

(c) (4 points) What point will the steepest descent method converge to, when applied to the function $f(x, y)$ with initial point $\mathbf{x}_0$?

(d) (6 points) Where will the steepest descent method be after one complete line search from $\mathbf{x}_0$?