CS296.1- Homework 5

Due on December 2, 2003

Questions may continue on the back. Please write clearly. What I cannot read, I will not grade. Typed homework is preferable. A good compromise is to type the words and write the math by hand.

1. If two matrices A and B commute, that is, if $AB = BA$, then

$$e^{A+B} = e^A \, e^B \; . \tag{1}$$

This exercise will take you through the proof of this fact.

(a) Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \; .$$

Show that A and B do not commute, and compute the matrices $e^A \, e^B$, $e^B \, e^A$, and e^{A+B}. This should convince you that equation (1) should not be taken for granted. Give exact answers, not numerical approximations.

(b) Prove by induction that if $AB = BA$ then B commutes with every power of A.

(c) Prove that B commutes with e^{At} for every real number t.

(d) Our main theorem is proven once we show that the matrix difference

$$F(t) = e^{(A+B)t} - e^{At} \, e^{Bt} \tag{2}$$

is zero as soon as A and B commute. To this end, prove first that if A and B commute, then $F(t)$ satisfies the differential equation

$$\dot{F}(t) = (A + B) \, F(t) \; . \tag{3}$$

(e) For the last step, we assume to be known from calculus that given any two $n \times n$ constant matrices A and B, the only $n \times n$ matrix solution $F(t)$ to the initial value problem

$$\dot{F}(t) = A \, F(t) \quad \text{with initial condition} \quad F(0) = B$$

is

$$F(t) = e^{At} B \; .$$

Using this fact, prove that if A and B commute then equation (1) is true. [Hint: what is $F(0)$ with $F(t)$ as defined in equation (2)?]

2. A matrix A is said to be *skew-Hermitian* if

$$A = -A^H \; .$$

Show that if A is skew-Hermitian then the $n \times n$ matrix

$$B = A^2 + cI \; ,$$

where c is a real scalar and I is the identity matrix, admits n orthonormal eigenvectors.

3. How can you find the Schur decomposition of the matrix

$$A = \begin{bmatrix} 3 & 2 & -4 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

and its eigenvalues without using any numerical code?

4. Solve the system of differential equations

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \mathbf{x}$$

with initial conditions $\mathbf{x}(0) = [1\ 0]^T$. You may leave the answer as a product of matrices, but all the entries of these matrices must be determined.