Polynomials

Polynomials have an important role in numerical computation for the following reasons.

- The evaluation of polynomials involves only arithmetic operations, which can be done on today’s digital computers.

- There are two important theorems about polynomials:
  - Taylor’s theorem, \(^1\)
  - Weierstrass approximation theorem. \(^2\)

  (Make some comparisons in their implications.)

We consider polynomials with real coefficients and real variable. Let \( P_n \) be the set of all polynomials of degree no larger than \( n \geq 0 \). Any polynomial in the set can be expressed as follows

\[
p_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,
\]

where \( a_j \in \mathcal{R} \) is the coefficient associated with the monic monomial \( x^j \), \( j = 0 : n \).

**Review of the basic polynomial properties**

- The set \( P_n \) is a vector space. In particular, it is closed with variable translation (shift of origin).
- The monic monomials
  \[
  1, x, x^2, \cdots, x^n,
  \]
  form a natural basis of \( P_n \). (Try to find another set of basis functions.) With a fixed set of basis functions, \( P_n \) can be mapped to the vector space \( \mathbb{R}^{n+1} \).

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\(^1\)Taylor, Brook, 1685-1731, England. First mathematics paper in 1708 on the center of oscillation of a body.

\(^2\)Weierstrass, Karl, 1815-1897, Germany. A high school teacher until well known for his original work.
Any \( p \in P_n \) is bounded on finite intervals, unbounded on infinite intervals.

Any polynomial in \( P_n \) is continuous and differentiable for arbitrarily many times, i.e., \( P_n \subset C^\infty \).

Differentiation is a linear operator (many-to-one projection) from \( P_n \) to \( P_{n-1} \); indefinite integration with constant zero is a linear operator from \( P_n \) to \( P_{n+1} \); and a definite integration is a linear operator (functional) from \( P_n \) to \( R \).

Any \( p_n(x) \) has \( n \) roots \( x_i \) in the complex plane, and can be represented in the factored form

\[
p_n(x) = \prod_{i=1}^{n} (x - x_i).
\]

Additional Note.

- There are other types of polynomials:
  - complex variable \( p(z), z \in C \)
    - special case \( z = e^{iz} \)
  - multiple variates \( p(x, y) \)

- Important applications of polynomials include
  - Approximate function evaluation by Taylor polynomials
  - Interpolation or data fitting with cubic splines
  - Multi-scale approximation with B-splines