CPS102- Homework 3

Due on October 20, 2005

Questions may continue on the back. Please write clearly. What I cannot read, I will not grade. Typed homework is preferable. A good compromise is to type the words and write the math by hand.

The Duke Community Standard requires every undergraduate student to sign the statement below upon completion of each academic assignment. I am not allowed to accept your assignment unless you sign on the line below, if you intend to return this sheet, or you copy and sign the same statement on your own paper.

I have adhered to the Duke Community Standard in completing this assignment.

Signature: ______________________________

In all answers, show your work in detail. The first two problems are from the book: number 26 on page 109 and number 34 on page 167.

1. If $f : B \to C$ and $f \circ g : A \to C$ are one-to-one, does it follow that $g : A \to B$ is one-to-one? Justify your answer. [Hint: your book has a solution for number 27, which has the same flavor, although different proof techniques may be needed. You may want to draw A, B, C and arrows, just to make sure you understand where the functions are defined.]

2. Show that if a is an integer and d is an integer greater than 1, then the quotient and remainder obtained when a is divided by d are $\lfloor a/d \rfloor$ and $a - d\lfloor a/d \rfloor$, respectively.

3. Functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are defined as follows:

$$f(x) = x^2 \qquad \text{and} \qquad g(x) = x^2 - 1 \; .$$

(a) Find the function $f \circ f$.

(b) Find the function $f \circ g$.

(c) Find the function $g \circ f$.

(d) Find the function $g \circ g$.

(e) List the elements of the set $\{x \in \mathbb{R} \mid (f \circ g)(x) = (g \circ f)(x)\}$.

4. Functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are defined as follows:

$$f(x) = \begin{cases} x+2 & \text{if } x < -1 \\ -x & \text{if } -1 \le x \le 1 \\ x-2 & \text{if } x > 1 \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} x-2 & \text{if } x < -1 \\ -x & \text{if } -1 \le x \le 1 \\ x+2 & \text{if } x > 1 \end{cases}$$

[Hint: sketch the functions.]

(a) Give an expression and a plot for the function $f \circ g$.

(b) Give an expression and a plot for the function $g \circ f$.

(c) Is f the inverse of g? Why or why not?

(d) Is g the inverse of f? Why or why not?

(e) Is f either injective or surjective? Explain.

(f) Is g either injective or surjective? Explain.

5. Let us use modulo arithmetic to prove that every year has at least one Friday the 13^{th}.

For any given year, let us number the days of the week with the integers from 0 to 6, assigning number 0 to the day of the week of January the 13^{th} that year, and then listing week days in chronological order. So the meaning of "day 0" depends on the year. In 2005, day 0 is Thursday, because January 13 was a Thursday.

(a) Write a table with twelve rows and five columns. The first column has the names of the months of the year. The second column is the number of days in each month *modulo 7* for a non-leap year. The third column is the the number of the week day corresponding to the 13^{th} of that month, again for a non-leap year, with the third-column entry for January set to 0. For instance, the third-column entry for February is 3. Forth and fifth column repeat columns 2 and 3 but for leap years (so columns 2 and 4 differ in only one place).

(b) In computing the third column of the table above, you should never have seen any number greater than 9 anywhere in your calculations. How did you compute or would you have computed column 3 in your table, given this constraint?

(c) In the same spirit, what is the fastest way to compute column 5?

(d) Prove that every year has a Friday the 13^{th}.

(e) Say that a year is *unlucky* if Friday the 13^{th} occurs three times that year. In what months does Friday the 13^{th} occur in unlucky leap and non-leap years?

(f) How often is a year unlucky? You may assume that every fourth year is a leap year (This is not quite true in reality, but simplifies your reasoning). However, you may not ignore leap years. Your answer should be in the form "exactly x out of any sequence of y consecutive years are unlucky." Show your reasoning in detail. [Hint: the answer would be very simple without leap years. To answer correctly in the presence of leap years, you need to consider enough years to make sure that your answer does not depend on where you start counting. The table from part (a) above is also useful. Take your time with this problem, and reason carefully one step at at time. To get started, what are the values of $365 \mod 7$ and $366 \mod 7$? What does that tell you about calendars of consecutive years?]