

## CPS102- Homework 5

Due on December 8, 2005

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Questions may continue on the back. Please write clearly. What I cannot read, I will not grade. Typed homework is preferable. A good compromise is to type the words and write the math by hand.

The Duke Community Standard requires every undergraduate student to sign the statement below upon completion of each academic assignment. I am not allowed to accept your assignment unless you sign on the line below, if you intend to return this sheet, or you copy and sign the same statement on your own paper.

*I have adhered to the Duke Community Standard in completing this assignment.*

Signature: \_\_\_\_\_

In all answers, show your work in detail.

1. In this problem we address the following question, posed by Alan Tucker:

In how many ways can a four-member committee with at least two women be formed from four men and six women, and so that Mr. John Baggins and Mrs. Samantha Baggins, both among the candidates, will not serve together?

Of course, people are distinguishable. For instance, two committees with four women are different if they are composed of different women.

Since this problem is tricky (as most combinatorial problems are), we will solve it in two different ways, and verify that we obtain the same solution.

- (a) Let  $C_m$  be the set of committees with  $m$  men, for  $m = 0, 1, 2$ , ignoring the rule about the Bagginses. What are the cardinalities  $\#(C_m)$  for each value of  $m$ ? Answer with both a general formula in terms of  $m$  and actual numbers.
- (b) For each value of  $m = 0, 1, 2$ , let  $V_m$  be the set of committees with  $m$  men that violate the rule about the Bagginses. Find the cardinalities  $\#(V_m)$ . Again, use a formula and actual numbers in your answer.
- (c) Assemble your answer to the committee problem from the cardinalities you found in your previous answers. When doing so, state clearly the reasons why you can add or subtract any values you add or subtract. Use the expression  $K_m$  for the number of valid committees with  $m$  men, and  $K$  for the total number of valid committees. A committee is *valid* if it satisfies all conditions of the problem.
- (d) As a sanity check, and as an exercise on a different method for counting, compute  $K$  by the counting method for sets of structured elements described in Section 3 of the handout on counting. To this end, determine first the number  $O$  of committees you could form if the order in which the committee members are chosen were to matter (*i.e.*, two committees that differ only by the order in which the members are chosen are temporarily assumed to be different committees). This will give you a greater number of committees than in your previous answer. In the question after this, you will be asked to correct for this over-counting.

Thus, an ordered committee is made of  $P = 4$  parts: member 1, member 2, member 3, and member 4. For each part, a member is one of four choices, whenever possible:  $M$  for “male but not Mr. Baggins,”  $F$  for “female but not Mrs. Baggins,”  $J$  for “Mr. John Baggins,” or  $S$  for “Mrs. Samantha Baggins.” Build a labelled tree of depth 4 as described in the handout, and compute the number  $O$  of ordered committees from it.

Make sure that you do not draw dead ends (subtrees that cannot lead to valid committees). Show your tree and your math in detail. Because of the (at most) four options for each part, the tree grows wide. Please redraw the tree until you get a decent drawing, don’t just hand in a bowl of spaghetti. It is OK to break the tree into subtrees and display the subtrees separately if you wish. In that case, label the subtree roots (say  $S_1$ , etc), and use the labels to show *clearly* where each subtree is to be appended. *Warning: this exercise requires patience.*

- (e) In the previous question, committees with the same people listed in different orders were considered as different committees. How many times is each unordered committee counted in the set of ordered committees? In other words, by what factor is  $O$  greater than  $K$ ?
- (f) Determine  $K$  from  $O$ , and check that the answer is the same you obtained earlier with the set composition method. If you obtain different answers, check and revise your work until the two numbers agree. Counting unordered committees by counting ordered ones first is useful as a sanity check. Hopefully, this exercise will also make you appreciate the importance of choosing a counting method that is appropriate for a given problem.

**2.** Let  $E_k$  be the event “the outcome of rolling two fair dice is  $k$ ” for  $k = 2, 3, \dots, 12$  (the “outcome” is the sum of the two numbers on the top faces of the two dice after rolling).

- (a) What is the sample space  $S$  for the roll of two dice, and what is the cardinality of  $S$ ?
- (b) Find  $p_k$ , the probability of  $E_k$ , for  $k = 2, 3, \dots, 12$ .
- (c) Do you expect the values of  $p_k$  to add up to 1? If yes, state why and check that they do. If not, explain why not.
- (d) The value of  $k$  associated with event  $E_k$  is a random variable with distribution  $p_k$ . Compute the expected value (or mean)  $m$  and variance  $\sigma^2$  of this random variable.
- (e) What is the conditional probability of the events  $E_k$ , defined earlier, given that the outcome of the first die is 1? Give the numerical value of the conditional probability for each  $k$  in  $\{2, \dots, 12\}$ .