Part I. Review: Pre-calculus, Calculus & Linear Algebra

A trigonometric polynomial \( f(x) \) of degree \( n, n \geq 0 \), may be expressed as follows,
\[
f(x) = a_0 + \sum_{k=1}^{n} a_k \cos(kx) + \sum_{k=1}^{n} b_k \sin(kx).
\]

Let \( T_n \) be the set of trigonometric polynomials of degree equal to or less then \( n, n > 0 \).

1. Verify that \( T_n \) is a vector space.

2. Verify that the set \( \{1/2, \cos(kx), \sin(kx), k = 1 : n\} \) forms a basis for \( T_n \). Determine the dimension of \( T_n \).

3. Because all functions in \( T_n \) are periodic with period \( 2\pi \), it suffices to consider \( T_n(-\pi, \pi) \) only.

   Define an inner product on \( T_n(-\pi, \pi) \) as follows
   \[
   \langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx.
   \]

   Verify that the above basis is orthogonal with respect to the inner product.

4. Let \( f \in T_n \). Verify that the coefficients of \( f \) can be analytically expressed as follows
   \[
a_k = \frac{1}{\pi} \langle \cos(kx), f(x) \rangle, \quad b_{k'} = \frac{1}{\pi} \langle \sin(k'x), f(x) \rangle, \quad k = 0 : n, k' = 1 : n.
   \]

5. Let \( f \in T_n \). Select a set of sample points. Describe the system of linear equations to determine the coefficients from the sampled data, assuming no noise in data.

Part II. Image compression with DCT

1. Provide a function to perform 2-d DCT and IDCT blockwise, respectively.

2. Provide a function for truncation and weighting of DCT coefficients.

3. Provide a function to evaluate the loss in a DCT compression.

4. Provide a script for demonstration:

   Note. Two project templates for function and script are provided.