Lecture 12: Computing Voronoi diagrams

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12.1 Introduction

In this lecture we'll discuss how to compute a Voronoi diagram and this will inolve primarily two new concepts:

- Lower Envelopes
- Duality

12.2 Lower Envelope

Suppose $F = \{f_1, f_2, \ldots, f_n\}$ be a set of functions where each $f_i : \mathbb{R}^d \to \mathbb{R}$. Lower envelope of F [Figure 12.1] is defined as,

$$L_F(\mathbf{x}) = min_{1 \le i \le n} f_i(\mathbf{x})$$

Similarly, we can define upper envelope of F as

$$U_F(\mathbf{x}) = max_{1 \le i \le n} f_i(\mathbf{x})$$

The points in the envelope(lower or upper) where one switches from one function to another is called a *breakpoint*[Figure 12.1]. For example in two dimensions when $y_i = f(x_i)$, if we project these breakpoints down onto the x-axis, we'll end up with a partition of the x axis. Note that each partition corresponds to only one function.

12.2.1 Minimization Diagrams

A concept related to lower envelopes is the concept of *minimization diagrams*. Formally, minimization diagram is the projection of the graph of L_F onto $\mathbb{R}^d$[Figure 12.1]. For a bivariate function($d = 2$), it is a planar subdivision.

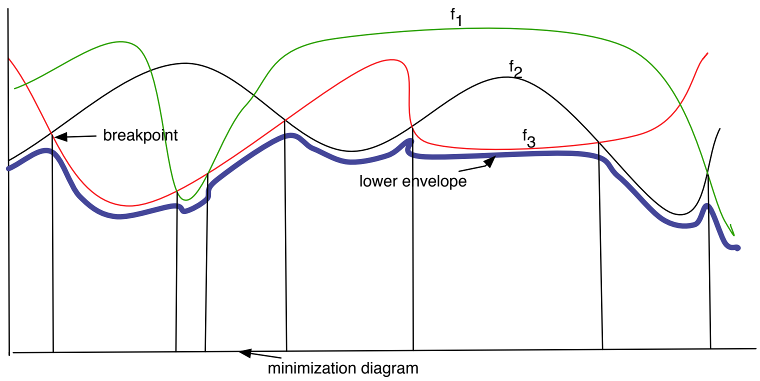

Figure 12.1: Lower Envelope and Minimization Diagram

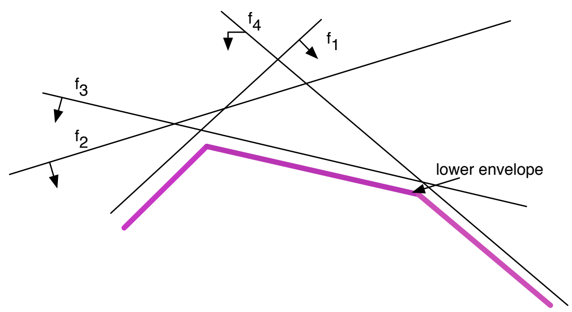

Figure 12.2: Lower Envelope for linear functions

12.2.2 Linear Functions and Lower Envelopes

Let our function f from the previous section be linear. Thus,

$$f : x_{d+1} = a_0 + a_1 x_1 + \ldots + a_d x_d$$

Let h_i be the halfspace lying below f_i. In other words, we consider the halfspaces where

$$x_{d+1} \leq a_0 + a_1 x_1 + \ldots + a_d x_d$$

Note that f_i be a linear function $\Rightarrow L_F = \partial(\cap_{i=1}^{n} h_i)$[Figure 12.2]
When we project L_F onto a plane we get a convex subdivision of the plane.
Let $S = \{p_1, p_2, \ldots, p_n\} \subseteq \mathbb{R}^2$. Let $p_i = \{x_i, y_i\}$.
Now define a function

$$f_i(\mathbf{x}) = ||\mathbf{x}p_i||$$

or

$$f_i(x, y) = \sqrt{(x - x_i)^2 + (y - y_i)^2}$$

.
Define

$$g_i(x, y) = f_i^2(x, y) - x^2 - y^2$$

. Thus,

$$\begin{aligned} g_i(x, y) &= (x - x_i)^2 + (y - y_i)^2 - x^2 - y^2 \\ &= -2xx_i - 2yy_i + x_i^2 + y_i^2 \end{aligned}$$

Note that $g_i(x, y)$ is a linear function. Let us denote the point with (x, y) coordinate as z.

Theorem 1 *Voronoi Diagram of S or Vor(S) is the minimization diagram of $G = \{g_1, g_2, \ldots, g_n\}$*

Proof: Suppose a point $z \in \mathbb{R}^2$ appears in a cell of the minimization diagram of G labeled i. Then,

$g_i(z) = min_{1 \leq j \leq n} g_j(z)$

$\Rightarrow g_i(z) \leq g_j(z), \forall j$

$\Rightarrow g_i(z) + x^2 + y^2 \leq g_j(z) + x^2 + y^2, \forall j \forall x \forall y$

$\Rightarrow f_i(z) \leq f_j(z)$

■

Complexity of Voronoi Diagram In two dimensions, the Voronoi Diagram has linear complexity while in three dimensions it has quadratic complexity. In general, size of the Voronoi Diagram in d dimensions, is $O(n^{\lceil \frac{d}{2} \rceil})$ where n is the number of sites. Note that instead of L_2 metric, we could have used the more general L_p [1] metric in defining $f_i(x, y)$. However, size complexity of the Voronoi Diagram for several metric is still an open question.

12.2.3 Geometric Interpretation of Voronoi diagram as minimization diagram

So what does $g_i(z)$ means geometrically? We can interpret $z = f_i(x, y)$ as a cone rooted at the point p_i. When we consider $z = f_i^2(x, y)$ instead, this cone turns into a paraboloid for each point p_i. So for the whole set S, we'll have a set of paraboloids, each rooted at a p_i, for $p_i \in S$. Initially we have a two dimensional plane P containing all the points in S. Recall that $g_i(x, y) = f_i^2(x, y) - x^2 - y^2$. So to account for $-(x^2 + y^2)$, we can take P and bend it so that it takes the form of an unit paraboloid $z = -(x^2 + y^2)$. Locate the point(say q_i) on this paraboloid which is vertically below $p_i, \forall i$. Draw a tangent plane $T(p_i)$ at q_i. Repeat this for all $i = 1$ to n. Denote this set of tangent planes by $H(S)$. Consider the convex polyhedron defined by the intersections of all the positive half-spaces defined by the set of $T(p_i)$s. If we project the edges and the vertices of the polyhedron vertically upwards onto the xy plane we'll get the Voronoi diagram of S[Figure 12.3].

[1] L_p metric for any two points a and b: $d_p(x, y) = (|a_x - b_x|^p + |a_y - b_y|^p)^{\frac{1}{p}}$

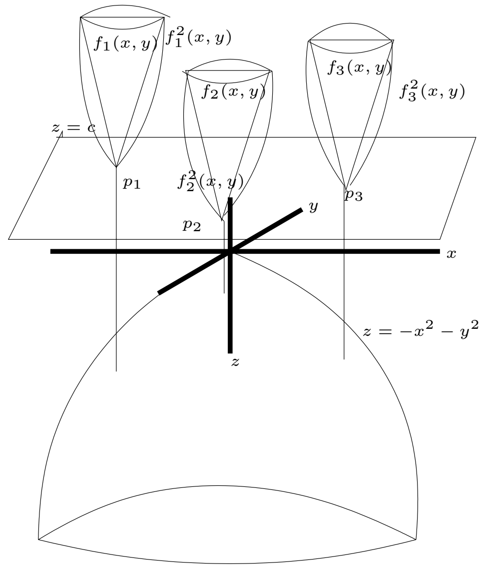

Figure 12.3: Goemetric Interpretation of Voronoi diagram as minimization diagram

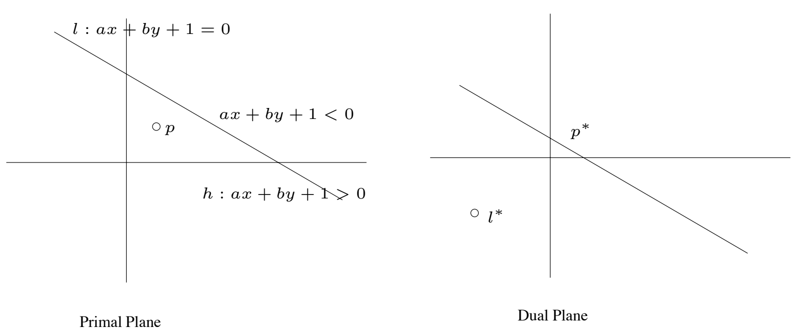

Figure 12.4: Example of Duality Transform

12.3 Duality

12.3.1 Definition

Duality is a transform that maps a point to a line and a line to a point. Note that we'll discuss duality in two dimensions but it works for higher dimensions also. Also we'll work with homogenous coordinates. Recall that for a point $p \in \mathbb{R}^2$, with coordinates $(\frac{x}{t}, \frac{y}{t})$ then the homogenous coordinates for p is (x, y, t). For simplicity we'll take $t = 1$. So for example if $a_1x_1 + a_2x_2 + a_3x_3 = 0$ is the equation of a line l, then by duality transform, the coordinates of the dual point p is (a_1, a_2, a_3). By homogenous coordinates transformation, $l : \frac{a_1}{a_3}x_1 + \frac{a_2}{a_3}x_2 + x_3 = 1, p : (\frac{a_1}{a_3}, \frac{a_2}{a_3})$. In general[Figure 12.4], if

$$l :< a, x >= 0 \Leftrightarrow l^* : a$$

and

$$p : (a, b) \Leftrightarrow p^* : ax + by + 1 = 0$$

12.3.2 Properties of Point Line Duality

Some of properties of duality transform are listed below:

1 **Containment**: If $p \in l \Leftrightarrow l^* \in p^*$. For instance, if p is (α, β) and l is $ax + by + 1 = 0$, then $a\alpha + b\beta + 1 = 0$. Similarly, if $l^* : (a, b)$ and $p^* : \alpha x + \beta y + 1 = 0$, then $a\alpha + b\beta + 1 = 0$.

2 **Self Inverse**:$(p^*)^* = p$ and $(l^*)^* = l$

3 **Order Reversing**: Point p lies above/below line l in the primal plane iff line p^* lies below/above point l^* in the dual plane respectively. In terms of halfplane, if $h : ax + by + 1 > 0$ is the halfplane defined by l containing the origin, and $p \in h$, then l^* has to lie in the halfspace h' of p^* containing the origin[Figure 12.4].

4 **Intersection preserving** Lines l_1 and l_2 intersect at point p iff line p^* passes through points l_1^* and l_2^* in the dual plane[Figure 12.5].

5 **Collinearity/coincidence** Three points are collinear in the primal plane, iff their dual lines intersect in a common point.

Observation The duality transform can also be applied to other objects than points and lines[1]. For instance the dual s^* of a line segment $s = \overline{pq}$ is the union of the duals of all the points on s. What we get is an infinite set of lines. All the points on s are collinear, so all the dual lines pass through one point. Their union forms a double wedge which is bounded by the duals of the endpoints of s. The lines dual to the endpoints of s define two double wedges, a left-right wedge and a top-bottom wedge; s^* is the left-right wedge[Figure 12.6]. It also shows that a line l intersecting s, whose dual l^* lies in s^*. This is because any line that intersects s must have either p or q above it and the other point below it, so the dual of such a line lies in s^* by the order preserving property of the dual transform. This oberservation will be fairly helpful in the following section.

Figure 12.5: Line Intersection in Duality Transform

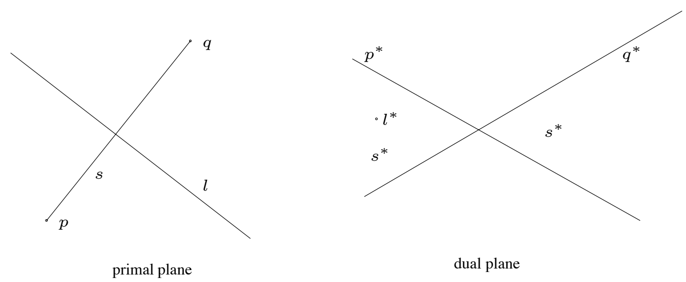

Figure 12.6: Duality Transform applied to a line segment

12.3.3 Duality and Convex Hull

Suppose we have a set L of n lines, $L = \{l_1, l_2, \dots, l_n\}$. Let h_i be the positive halfplane bounded by l_i. So $P = \cap_{i=1}^{n} h_i$ is a convex polygon. So $p \in P \Rightarrow p \in h_i, \forall i$. Look at the dual of these lines. Each l_i maps to a l_i^*. Note that $l_i^* \in (p^*)^+$ if $(p^*)^+$ is the positive halfplane bounded by p^*. Let us look at the convex hull of $L^*(conv(L^*))$. Observe that, if $p \in P$, then p lies outside $conv(L^*)$. In other words, $p^* \cap conv(L^*) = \emptyset$. Let q be the intersection point of l_i and l_j in P, then by property (4), q^* is an edge of $conv(L^*)$. In summary, $conv(L^*)$ is the dual of P [Figure 12.7]. Note that in three dimensions, a face of P maps to a vertex of $conv(L^*)$, an edge of P maps to an edge of $conv(L^*)$ and a vertex of P maps to a face of $conv(L^*)$.

Duality transform for unbounded polyhedron Let L be a set of lines, $L = \{l_1, l_2, \dots, l_n\}$ where l_i : $-2a_i x - y + a_i^2 = 0$. Thus , l_i^* is $(-\frac{2a_i}{a_i^2}, -\frac{1}{a_i^2})$. This means that in the dual plane, all the points lie below the x axis. However, note that if the convex hull does not contain the origin it will be unbounded.

Theorem 2 *Let P be a set of points in the plane. The counterclockwise order of the points along the upper (lower) convex hull of P , is equal to the left to right order of the sequence of lines on the lower (upper) envelope of the dual P^* [2].*

Proof: We will prove the result just for the upper hull and lower envelope, since the other case is symmetrical. For simplicity, let us assume that no three points are collinear. Observe that a necessary and sufficient condition for a pair of points p_i and p_i to form an edge on the upper convex hull is that the line e(ij) that passes through both of these points has every other point in P strictly beneath it.

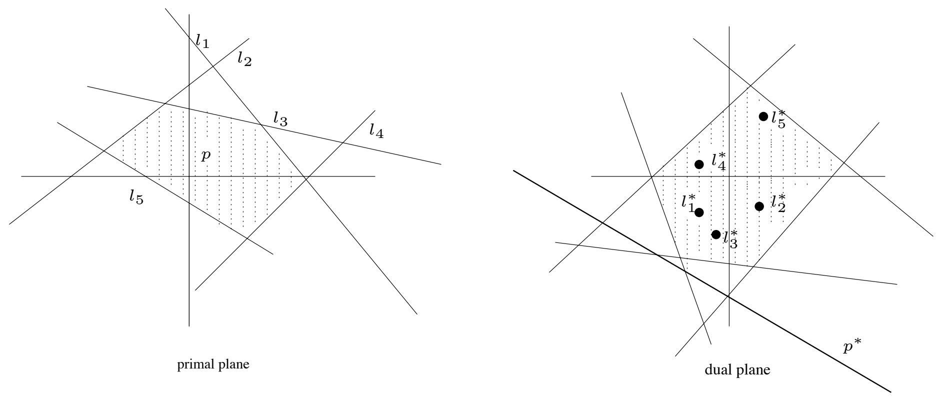

Figure 12.7: Duality and convex hulls

Consider the dual lines p_i^* and p_j^*. A necessary and sufficient condition that these lines are adjacent on the lower envelope is that the dual point at which they meet, l_{ij}^* lies beneath all of the other dual lines in p^*. The order reversing condition of duality assures us that the primal condition occurs if and only if the dual condition occurs. Therefore, the sequence of edges on the upper convex hull is identical to the sequence of vertices along the lower envelope. As we move counterclockwise along the upper hull observe that the slopes of the edges increase monotonically. Since the slope of a line in the primal plane is the a coordinate of the dual point, it follows that as we move counterclockwise along the upper hull, we visit the lower envelope from left to right. ■

References

[1] Berg M.de, Kreveld M.van, Overmars M. Schwarzkopf O *Computational Geometry Algorithms and Applications*

[2] http://www.cs.wustl.edu/ pless/506/l8.html