CPS102- Homework 2

Due on September 29, 2006

Questions may continue on the back. Please write clearly. What I cannot read, I will not grade. Typed homework is preferable. A good compromise is to type the words and write the math by hand. Show all your work in detail.

The Duke Community Standard requires every undergraduate student to sign the statement below upon completion of each academic assignment. I am not allowed to accept your assignment unless you sign on the line below, if you intend to return this sheet, or you copy and sign the same statement on your own paper.

I have adhered to the Duke Community Standard in completing this assignment.

Signature: ______________________________

Important: A fundamental component of this course is about thinking formally and making clear arguments. Inaccurate, incomplete, or poorly phrased arguments will not receive full credit.

1. The following axioms

$$\forall x\, (P(x) \lor Q(x))\,,\ \exists x \neg Q(x)\,,\ \forall x\, (R(x) \rightarrow \neg P(x))$$

entail the following formula ϕ:

$$\exists x \neg R(x)\ .$$

Prove ϕ from the given axioms *by resolution with refutation.* Please take this directive seriously: no credit will be given for proofs of a different style (e.g., giving some intuitive argument, or using truth tables, or using resolution without refutation). Show all your substitutions and resolvents, and state which literals you are resolving away at each step.

The following problem is number 24 on page 131 of the book (sixth edition). The one thereafter is a modification of problem 40 on the same page.

2. Let A, B, and C be sets. Show that

$$(A - B) - C = (A - C) - (B - C)\ .$$

[Hints: Set difference is defined in the book. To solve this problem, show that each side of the equation is contained in the other (a "mutual inclusion" proof). To prove inclusion of set S in set T means to show that every x that is in S is also in T:

$$\forall x\ (x \in S \ \rightarrow \ x \in T)\ .$$

To "show" means to give a clear, well-supported argument.]

3. The *symmetric difference* of A and B, denoted by $A \oplus B$, is the set containing those elements in either A or B, but not in both A and B. In other words, an element in $A \oplus B$ is either in A but not in B, or in B but not in A:

$$A \oplus B = (A \cap \overline{B}) \cup (\overline{A} \cap B)\ .$$

Show that the symmetric difference is associative:

$$A \oplus (B \oplus C) = (A \oplus B) \oplus C\ .$$

Explain your reasoning carefully. [Hint: Either draw a Venn diagram for each side or go through blind formula manipulation. The second method is more straightforward but very boring and error prone, the first is more interesting. If you choose the boring method, you get no credit if you make mistakes in the manipulation. If you use Venn diagrams, make sure that you explain clearly how you use them, and how your drawings support your conclusion.]

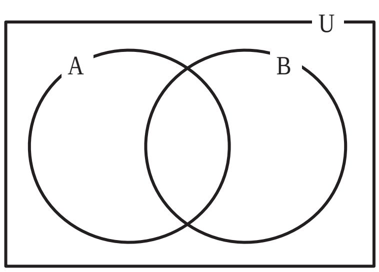
4. Let A be the set of prime numbers smaller than 10, and B the set of perfect squares less than 20. [Warning: 1 is not prime, and 0 is a perfect square.] Your answers below should be in the form of *exact integers*.

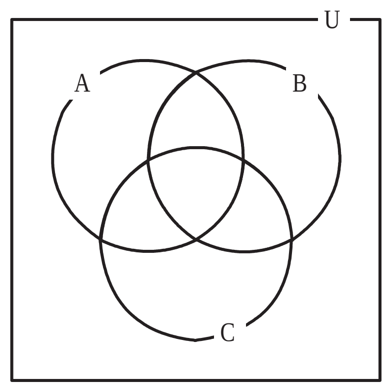
(a) How many elements are in the power set of the Cartesian product of A and B?

(b) How many elements are in the Cartesian product of the power sets of A and B?

(c) How many elements are in the power set of the power set of B?

5. A Venn diagram for n sets $A_1, \ldots, A_n$ is *complete* if it has a region of nonzero area for every *component*, that is, for every set of the form $L_1 \cap \ldots \cap L_n$ where L_i is either A_i or its complement, and '$\cap$' is set intersection. For instance, the components for $n = 2$ are AB, $A\overline{B}$, $\overline{A}B$, and $\overline{A}\,\overline{B}$. Here are complete Venn diagrams for two and three sets, respectively:





(a) Make a reasonably large copy of the complete diagram for three sets A, B, C (figure on the right above), and label each of its eight components. For instance, the region that is in A and B but not in C is labeled $A \cap B \cap \overline{C}$. You may abbreviate this to $AB\overline{C}$. Place the label in the interior of each component.

(b) How many components does a complete Venn diagram for n sets have? Prove it.

(c) Draw a complete Venn diagram for $n = 4$ sets and label all the components. *Make a clean drawing. No credit for unintelligible drawings. This means that you probably need to make a few drafts on separate paper before you draw your final solution.* You get five extra-credit points if in your diagram (i) components are represented by *connected* regions, and (ii) no more than two boundaries cross each other at any point. For instance, the diagram for $n = 3$ below violates rule (i) because component $\overline{A}\,\overline{B}C$ is represented by two separate regions. This diagram also violates rule (ii) because three boundaries cross at one point.