## CPS102- Homework 5

Due on December 8, 2006

**Questions may continue on the back. Please write clearly. What I cannot read, I will not grade. Typed homework is preferable. A good compromise is to type the words and write the math by hand. Show all your work in detail.**

**The Duke Community Standard requires every undergraduate student to sign the statement below upon completion of each academic assignment. I am not allowed to accept your assignment unless you sign on the line below, if you intend to return this sheet, or you copy and sign the same statement on your own paper.**

*I have adhered to the Duke Community Standard in completing this assignment.*

Signature: _____

In all answers, show your work in detail.

**1**. This problem starts with a classical question in probability posed by the Duke of Tuscany to Galileo (in a slightly different form). In your answers to the following three questions, give **exact** numerical values, **both** as irreducible fractions of integers and as decimal numbers. For instance, a value of $p_9 = 3/18$ would be reported as

$$p_9 = \frac{1}{6} = 0.1\overline{6} \; .$$

You will lose credit for incomplete or inappropriate answer formats.

**(a)** Determine the probability $p_9$ of obtaining a total of 9 when three fair dice are tossed once, and the probability $p_{10}$ of obtaining a total of 10 when three fair dice are tossed once. Outcomes for the three dice are mutually independent. To answer this question you may need to build a table of possible cases, but keep this table small.

**(b)** Suppose now that two of the three dice are fair, but one is loaded. Specifically, the probability of obtaining a 1 with the loaded die is three times as great as the probability of obtaining any one of the other five values (that is, a 1 is three times more likely than a 2, three times more likely than a 3, and so forth). Write the probability distribution over the sample space for the roll of the loaded die (by itself).

**(c)** What are the probabilities $p_9$ and $p_{10}$ when two of the three dice are fair and the third is loaded as above?

**2**. Prove or disprove the following statement: If $E$ and $F$ are independent and $P(G) > 0$, then

$$P(E \cap F \mid G) = P(E \mid G)\, P(F \mid G) \; .$$

In words, if $E$ and $F$ are independent, they are also conditionally independent.

**3**. What can you say about $P(E)$ if $P(F) > 0$ and $P(E \mid F) = P(F)$ and $E$ and $F$ are independent?

**4**. Five percent of the people have high blood pressure. Three quarters of the people with high blood pressure drink alcohol. Half of the people without high blood pressure drink alcohol. What fraction of the alcohol drinkers have high blood pressure? Give a percentage with two accurate significant digits.

**5**. Let $X$ be a random variable with mean (a.k.a. expected value) $m$ and variance $\sigma^2$, and consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(y) = E[(X - y)^2] \; .$$

Show that the smallest value of $f(y)$ is achieved when $y = m$. This means that the mean of a random variable is its "best" estimate, in the sense that it is the number $y$ that has the smallest expected squared deviation $(X-y)^2$ from $X$. Interestingly, this fact does not depend on the distribution of $X$. [Hint: differentiation is a linear operation, and so is expectation.]

**6**. A fair coin is tossed until a head appears. Let $X$ denote the number of tosses required. Tosses are mutually independent.

**(a)** What is the sample space for this experiment? Think carefully: what is an individual outcome?

**(b)** What is the range of the random variable $X$?

**(c)** What is the probability distribution $p_X$ of $X$?

**(d)** Show that your $p_X$ is indeed a probability distribution.

**(e)** What is the expected number of tosses required for a head to appear? You may use formulae in Table 2 on page 157 of Rosen (6th edition; this table is on page 232 in the 5th edition).