Exercises

The credit assignment reflects a subjective assessment of difficulty. A typical question can be answered using knowledge of the material combined with some thought and analysis.

1. **Deciding isomorphism** (three credits). What is the computational complexity of recognizing isomorphic abstract simplicial complexes?
2. **Order complex** (two credits). A *flag* in a simplicial complex K in $\mathbb{R}^d$ is a nested sequence of proper faces, $\sigma_0 < \sigma_1 < \ldots < \sigma_k$. The collection of flags form an abstract simplicial complex A sometimes referred to as the *order complex* of K. Prove that A has a geometric realization in $\mathbb{R}^d$.
3. **Barycentric subdivision** (one credit). Let K consist of a d-simplex σ and its faces.
   (i) How many d-simplexes belong to the barycentric subdivision, $\mathrm{Sd}K$?
   (ii) What is the d-dimensional volume of the individual d-simplices in $\mathrm{Sd}K$?
4. **Covering a tree** (one credit). Let P be a finite collection of closed paths that cover a tree, that is, each node and each edge of the tree belongs to at least one path.
   (i) Prove that the nerve of P is contractible.
   (ii) Is the nerve still contractible if we allow subtrees in the collection? What about sub-forests?
5. **Nerve of stars** (one credit). Let K be a simplicial complex.
   (i) Prove that K is a geometric realization of the nerve of the collection of vertex stars in K.
   (ii) Prove that $\mathrm{Sd}K$ is a geometric realization of the nerve of the collection of stars in K.
6. **Helly for boxes** (two credits). The *box* defined by two points $a = (a_1, a_2, \ldots, a_d)$ and $b = (b_1, b_2, \ldots, b_d)$ in $\mathbb{R}^d$ consists of all points x whose coordinates satisfy $a_i \leq x_i \leq b_i$ for all i. Let F be a finite collection of boxes in $\mathbb{R}^d$. Prove that if every pair of boxes has a non-empty intersection then the entire collection has a non-empty intersection.
7. **Alpha complexes** (two credits). Let $S \subseteq \mathbb{R}^d$ be a finite set of points in general position. Recall that Čech(r) and Alpha(r) are the Čech

and alpha complexes for radius $r \geq 0$. Is it true that $\text{Alpha}(r) = \check{\text{C}}\text{ech}(r) \cap \text{Delaunay}$? If yes, prove the following two subcomplex relations. If no, give examples to show which subcomplex relations are not valid.

(i) $\text{Alpha}(r) \subseteq \check{\text{C}}\text{ech}(r) \cap \text{Delaunay}$.

(ii) $\check{\text{C}}\text{ech}(r) \cap \text{Delaunay} \subseteq \text{Alpha}(r)$.

8. **Collapsibility** (three credits). Call a simplicial complex *collapsible* if there is a sequence of collapses that reduce the complex to a single vertex. The existence of such a sequence implies that the underlying space of the complex is contractible. Describe a finite 2-dimensional simplicial complex that is not collapsible although its underlying space is contractible.