Exercises

The credit assignment reflects a subjective assessment of difficulty. A typical question can be answered using knowledge of the material combined with some thought and analysis.

1. **Tetrahedron complex** (one credit). Let K consist of a tetrahedron and its faces.
   (i) Apply the matrix reduction algorithm to the filtration of K obtained by adding the simplices in the order of dimension.
   (ii) Do any of the three diagrams depend on the way you order the simplices of the same dimension?
2. **Betti numbers of alpha complexes** (three credits). Apply the incremental Betti number algorithm to the sequence of alpha complexes of a finite set of points in $\mathbb{R}^3$. Let n be the number of simplices in the last complex, the Delaunay triangulation.
   (i) Use the union-find data structure to maintain the components of the algorithm and argue that this gives the zeroth Betti numbers of all alpha complexes in time $\mathrm{O}(n\alpha(n))$, where α is the inverse of the fast growing Ackermann function.
   (ii) Show that the same algorithm can be used to maintain the components of the complement and thus get the second Betti numbers in the same amount of time.
   (iii) Get the first Betti numbers from the zeroth and second Betti numbers using the Euler-Poincaré Theorem.
3. **Examples of switches** (two credits). Given examples for the types of switches analogous to the ones shown in Figure V.8 but one dimension up in each of the three types.
4. **Bipartite graph matching** (three credits). Given a bipartite graph with $n + n$ vertices, the algorithm by Hopcroft and Karp takes time $\mathrm{O}(n^{2.5})$ to decide whether or not it contains a perfect matching.
   (i) Let A and B be two sets of n points in $\mathbb{R}^2$ each. Use the Hopcroft-Karp algorithm as a subroutine to compute a perfect matching between A and B that minimizes the length of the longest edge in time $\mathrm{O}(n^{2.5} \log n)$.
   (ii) Adapt your algorithm to the case in which A and B are two persistence diagrams with possibly different numbers of points.

5. **Cauchy-Crofton** (two credits). Generalize the Cauchy-Crofton formula for curves in the plane given in Section V.3 to
   (i) curves in three-dimensional Euclidean space;
   (ii) surfaces in three-dimensional Euclidean space.
6. **Sublevel sets** (two credits). Let $f : K \to \mathbb{R}$ be a piecewise linear function defined by its values at the vertices, $f(u_1) < f(u_2) < \ldots < f(u_n)$. Let b be strictly between $f(u_i)$ and $f(u_{i+1})$, for some $1 \leq i \leq n-1$, and recall that the sublevel set defined by b is $f^{-1}(-\infty, b]$.
   (i) Prove that the sublevel sets defined by b and by $f(u_i)$ have the same homotopy type.
   (ii) Draw an example each for the cases when the sublevel sets defined by b and by $f(u_{i+1})$ have the same and different homotopy types.
7. **Persistence diagram** (one credit). Draw a genus-3 torus, consider its height function, and draw the non-trivial persistence diagrams of the function. Distinguish between points in the ordinary, extended, and relative sub-diagrams.
8. **Breaking symmetry** (two credits). Design a topological space $\mathbb{X}$ and a continuous function $f : \mathbb{X} \to \mathbb{R}$ such that
   (i) the persistence diagrams violate the Duality Theorem of Section V.4;
   (ii) the persistence diagrams violate the Symmetry Theorem of the same section.