IV.3 Matrix Reduction

In this section, we show how to compute Betti numbers of triangulated topological spaces. We also introduce the notion of relative homology which is useful in studying spaces that are not necessarily closed.

Boundary matrices. It is convenient to represent the boundary operator as a matrix that records the incidences between simplices. Using standard row and column operations we can extract the ranks of cycle and boundary groups and this way compute Betti numbers. Let K be a simplicial complex. Its *p-th boundary matrix* represents the $(p-1)$-simplices as rows and the p-simplices as columns. Assuming an ordering on the simplices of the same dimension, this matrix is $\partial_p = [a_i^j]$, where i ranges from 1 to n_{p-1}, j ranges from 1 to n_p, and $a_i^j = 1$ iff the i-th $(p-1)$-simplex is a face of the j-th p-simplex. Given an n_p-vector representing a p-chain, the boundary can be computed by matrix multiplication,

$$\partial_p c_p = \begin{bmatrix} a_1^1 & a_1^2 & \ldots & a_1^{n_p} \\ a_2^1 & a_2^2 & \ldots & a_2^{n_p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n_{p-1}}^1 & a_{n_{p-1}}^2 & \ldots & a_{n_{p-1}}^{n_p} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n_p} \end{bmatrix}.$$

In words, a collection of columns represents a p-chain and the sum of these columns gives its boundary. Similarly, a collection of rows represents a $(p-1)$-chain and the sum of these rows gives its coboundary, a concept that will be defined in the next section.

Row and column operations. The rows of the matrix ∂_p form a basis of the $(p-1)$-st chain group, C_{p-1}, and the columns form a basis of the p-th chain group, C_p. We use two types of column operations to modify the matrix without changing its rank: exchanging columns k and l and adding column k to column l. Both can be expressed by multiplying with a matrix $V = [v_i^j]$ from the right. To exchange two columns we have $v_k^l = v_l^k = 1$ and $v_i^i = 1$ for all $i \neq k, l$. All other entries are zero. To add column k to column l we have $v_k^l = 1$ and $v_i^i = 1$ for all i. All other entries are zero. As indicated in Figure IV.7, the effect of the operation is that the l-th column now represents the sum of the k-th and the l-th p-simplices, or the sum of whatever the two columns represented before the operation. We have similar two row operations, one exchanging two rows and the other adding one row to another. This translates to multiplication with a matrix $U = [u_i^j]$ from the left. To exchange two rows

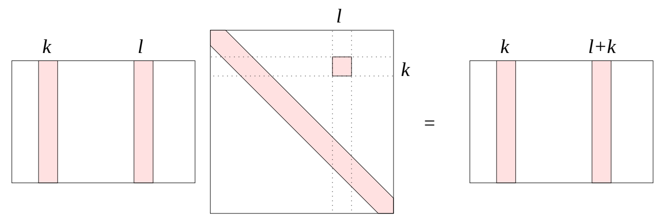


Figure IV.7: The effect of a single off-diagonal one in the matrix V is the addition of one column in the boundary matrix to another. The effect on the basis of C_p is similar.

we again have $u_k^l = u_l^k = 1$, $u_i^i = 1$ for $i \neq k, l$, and all other entries zero. To add the k-th to the l-th row we have $u_l^k = 1$, $u_i^i = 1$ for all i, and all other entries zero, as in Figure IV.8. The effect of this operation is that the k-th

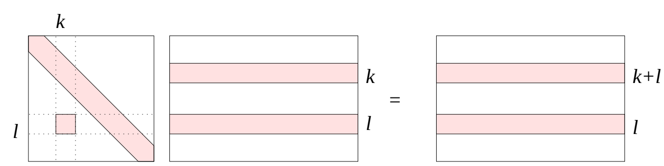


Figure IV.8: The effect of a single off-diagonal one in the matrix U is the addition of one row in the boundary matrix to another. The effect on the basis of C_{p-1} is that the row that was added now represents the sum of $(p-1)$-chains, which is symmetric to the effect of a column addition.

row now represents the sum of the k-th and the l-th $(p-1)$-simplices, or the sum of whatever the two columns represented before the operation. Although the $(p-1)$- and p-chains represented by the rows and columns change as we perform row and column operations, they always represent bases of the two chain groups.

Smith normal form. Using row and column operations we can reduce the p-th boundary matrix to *Smith normal form*. For modulo 2 arithmetic this means an initial segment of the diagonal is 1 and everything else is 0, as in Figure IV.9. Recall that $n_p = \operatorname{rank} \mathsf{C}_p$ is the number of columns of the p-th boundary matrix. Let $n_p = z_p = b_{p-1}$ such that the leftmost b_{p-1} columns have ones in the diagonal and the rightmost z_p columns are zero. The former represent p-chains whose non-zero boundaries generate the group of $(p-1)$-

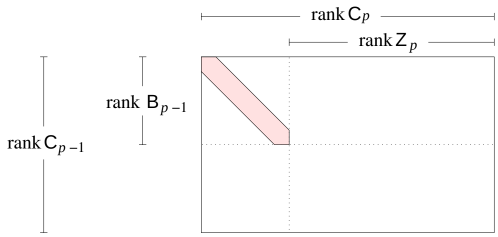


Figure IV.9: The entries in the shaded, initial portion of the diagonal are all 1 and all other entries are 0.

boundaries. The latter represent p-cycles that generate Z_p. Once we have all boundary matrices in normal form, we can extract the ranks of the boundary and cycle groups and get the Betti numbers,

$$\beta_p = \operatorname{rank} \mathsf{Z}_p - \operatorname{rank} \mathsf{B}_p$$

for $p \geq 0$. It is obvious now but useful to remember that computing β_p requires the ranks of two boundary matrices, not just one.

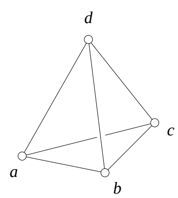
To reduce ∂_p we proceed similar to Gaussian elimination for solving a system of linear equations. In at most two exchange operations we move a 1 to the upper left position, and with at most $n_{p-1} - 1$ row and $n_p - 1$ column additions we zero out the rest of the first row and first column. We then recurse for the submatrix obtained by removing the first row and first column. Let x be the row and column number of the upper left element of the submatrix we consider. Initially, $x = 1$.

```
void Reduce(x)
  if ∃k ≥ x, l ≥ x with ∂_p[k, l] = 1 then
    exchange rows x and k;  exchange columns x and l;
    for i = x + 1 to n_{p−1} do
      if ∂_p[i, x] = 1 then add row x to row i endif
    endfor;
    for i = x + 1 to n_p do
      if ∂_p[x, j] = 1 then add column x to column j endif
    endfor;
    Reduce(x + 1)
  endif.
```

We have at most n_{p-1}^2 row operations and at most n_p^2 column operations. The total amount of time is therefore at most some constant times $n_{p-1} n_p (n_{p-1} +$

n_p). In summary, we reduce the boundary matrices in time at most cubic in the number of simplices in K. From the reduced matrices we readily get the Betti numbers.

Examples. As a first example consider the edge skeleton of the tetrahedron. The only non-trivial boundary matrix is ∂_1. We use row and column operations to reduce ∂_1 to Smith normal form, as shown in Figure IV.10. We conclude that



	ab	ac	ad	bc	bd	cd
a	1	1	1			
b	1			1	1	
c		1		1		1
d			1		1	1

	ab	ab ac	ac ad	ab ac bc	ab ad bd	ac ad cd
b a	1					
c b		1				
d c			1			
d						

Figure IV.10: From left to right: the edge skeleton of the tetrahedron, its first boundary matrix, the reduced boundary matrix with $\operatorname{rank} \mathsf{B}_0 = 3$ non-zero rows and $\operatorname{rank} \mathsf{Z}_1 = 3$ zero columns.

the edge skeleton has $\beta_0 = \operatorname{rank} \mathsf{Z}_0 - \operatorname{rank} \mathsf{B}_0 = 4 - 3 = 1$ and $\beta_1 = \operatorname{rank} \mathsf{Z}_1 - \operatorname{rank} \mathsf{B}_1 = 3 - 0 = 3$. All other Betti numbers are zero. If we add the four triangles to the complex we get another boundary matrix to reduce, as shown in Figure IV.11. The new matrix does not affect dimension zero so $\beta_0 = 1$, as

	abc	abd	acd	bcd
ab	1	1		
ac	1		1	
ad		1	1	
bc	1			1
bd		1		1
cd			1	1

	abc	abc abd	abc abd acd	abc abd acd bcd
bc ac ab	1			
cd bc ad ac		1		
cd bd bc			1	
ad				
bd				
cd				

Figure IV.11: The original boundary matrix, ∂_2, on the left and its reduced form with $\operatorname{rank} \mathsf{B}_1 = 3$ non-zero rows and $\operatorname{rank} \mathsf{Z}_2 = 1$ zero column on the right.

before. For dimensions one and two we have $\beta_1 = \operatorname{rank} \mathsf{Z}_1 - \operatorname{rank} \mathsf{B}_1 = 3 - 3 = 0$ and $\beta_2 = \operatorname{rank} \mathsf{Z}_2 - \operatorname{rank} \mathsf{B}_2 = 1 - 0 = 1$.

Reduced homology. Homology groups have been defined for triangulated spaces, which are therefore necessarily closed. To extend them to other spaces, we introduce homology groups for pairs of closed spaces. Let K be a simplicial complex and K_0 and subcomplex of K. The *relative chain groups* are quotients of the chain groups of K and of K_0, $\mathsf{C}_p(K, K_0) = \mathsf{C}_p(K)/\mathsf{C}_p(K_0)$. Taking this quotient partitions $\mathsf{C}_p(K)$ into cosets of p-chains that possibly differ in the p-simplices in K_0 but not in the ones in $K - K_0$. The *boundary operator* is induced by the one for K,

$$\partial_p : \mathsf{C}_p(K, K_0) \to \mathsf{C}_{p-1}(K, K_0).$$

As before, ∂ commutes with addition and $\partial \circ \partial = 0$. We thus define *relative cycle groups*, *relative boundary groups*, and *relative homology groups* as before,

$$\begin{aligned}
\mathsf{Z}_p(K, K_0) &= \ker(\partial_p : \mathsf{C}_p(K, K_0) \to \mathsf{C}_{p-1}(K, K_0));\\
\mathsf{B}_p(K, K_0) &= \operatorname{im}(\partial_{p+1} : \mathsf{C}_{p+1}(K, K_0) \to \mathsf{C}_p(K, K_0));\\
\mathsf{H}_p(K, K_0) &= \mathsf{Z}_p(K, K_0)/\mathsf{B}_p(K, K_0).
\end{aligned}$$

Let $c + \mathsf{C}_p(K_0)$ be a relative p-chain. It is a relative p-cycle iff ∂c is carried by K_0. Furthermore, it is a relative p-boundary is there is a $(p+1)$-chain d of K such that $c - \partial d$ is carried by K_0; see Figure IV.12.

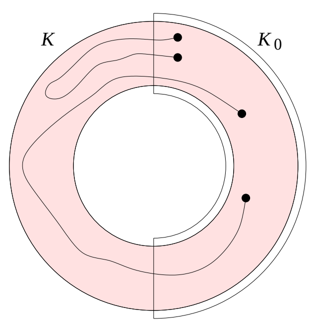


Figure IV.12: The two paths are neither boundaries nor cycles in K but they are both relative cycles and one is a relative boundary in (K, K_0).

Excision. By construction, relative homology depends only on the part of K outside K_0 and ignores the part inside K_0. Hence we can remove simplices from both complexes without changing the homology. To make this precise let L be the smallest subcomplex of K that contains $K - K_0$ and define $L_0 = L - (K - K_0)$. Since L contains $K - K_0$ we have $L - L_0 = K - K_0$.

EXCISION THEOREM. Let $K_0 \subseteq K$ and $L_0 \subseteq L$ be as defined above. Then the two sequences of relative homology groups are pairwise isomorphic, $\mathsf{H}_p(K, K_0) \simeq \mathsf{H}_p(L, L_0)$ for all p.

Instead of giving an algebraic proof of this fairly obvious fact, let us take a look at the algorithm for computing ranks of relative homology groups. Starting with the boundary matrices for K, we delete all rows and columns that correspond to simplices in K_0. Clearly, if we start with the boundary matrices for L and remove all rows and columns that correspond to simplices in L_0 we end up with the same matrices, namely the ones that records incidences between simplices in $K - K_0 = L - L_0$. Reducing these matrices gives the ranks of the boundary, cycle, and homology groups which are therefore the same for (K, K_0) and (L, L_0).

Bibliographic notes. We have introduced homology groups for modulo 2 arithmetic, but the matrix reduction algorithm suggests that other, more elaborate coefficient groups can also be used. Texts in algebraic topology focus on using integer coefficients, but this complicates matters significantly: we need to orient simplices so that negative multiples are defined, homology groups develop torsion that is not measured by their ranks, the reduction algorithm needs to compute common factors of numbers; see e.g. Munkres [1].

[1] J. R. MUNKRES. *Elements of Algebraic Topology.* Addison-Wesley, Redwood City, California, 1984.