Computer Vision

CPS 296.1 Supplementary Lecture Notes

Carlo Tomasi – Duke University

Fall 2007

6. Variational Optic Flow

[Last Modified on November 14, 2007]

Warning. These notes are being written as the course progresses. As errors are discovered, old notes will be corrected and the new version will be posted online.

6 Variational Optic Flow

B. P. K. Horn casts optic flow in terms of the a functional minimization problem of the following form [1]:

$$(\hat{u}, \hat{v}) = \arg\min_{u,v} \iint_D F(u, v, u_x, u_y, v_x, v_y)\, dx\, dy$$

where

$$u = u(x, y) \quad \text{and} \quad v = v(x, y)$$

and no conditions are given for the values of u or v on the boundary ∂D of D.

Here we prove the following theorem of the calculus of variations:

The solution $(\hat{u}, \hat{v})$ to the optic flow problem is a solution to the following *Euler equations*:

$$\begin{aligned} F_u - \frac{\partial}{\partial x} F_{u_x} - \frac{\partial}{\partial y} F_{u_y} &= 0 \\ F_v - \frac{\partial}{\partial x} F_{v_x} - \frac{\partial}{\partial y} F_{v_y} &= 0 \end{aligned}$$

with the following *natural boundary conditions*:

$$\begin{aligned} F_{u_x} \frac{dy}{ds} &= F_{u_y} \frac{dx}{ds} \\ F_{v_x} \frac{dy}{ds} &= F_{v_y} \frac{dx}{ds} \end{aligned}$$

where s is a parameter along the boundary ∂D of D.

Note that if D is the entire image, then the natural boundary conditions become simply

$$\begin{aligned} F_{u_x} = F_{v_x} = 0 &\qquad \text{on the vertical boundaries, and} \\ F_{u_y} = F_{v_y} = 0 &\qquad \text{on the horizontal boundaries.} \end{aligned}$$

To prove this theorem, we introduce two *test functions*

$$\alpha(x, y) \quad \text{and} \quad \beta(x, y)$$

and two real variables δ, ϵ, and let

$$I(u, v, \delta, \epsilon) = \iint_D F(u + \delta\alpha, v + \epsilon\beta, u_x + \delta\alpha_x, u_y + \delta\alpha_y, v_x + \epsilon\beta_x, v_y + \epsilon\beta_y)\, dx\, dy$$

be a *variation* of the original integral $I(u, v, 0, 0)$. Then the optic flow field (u, v) is a minimum for this functional if and only if

$$\left.\frac{\partial I}{\partial \delta}\right|_{\delta=\epsilon=0} = 0 \quad \text{and} \quad \left.\frac{\partial I}{\partial \epsilon}\right|_{\delta=\epsilon=0} = 0$$

for *all* test functions α and β.

If F is differentiable at least once, then

$$\begin{aligned}\frac{\partial I}{\partial \delta} &= \iint_D (\alpha F_u + \alpha_x F_{u_x} + \alpha_y F_{u_y})\, dx\, dy \\ \frac{\partial I}{\partial \epsilon} &= \iint_D (\beta F_v + \beta_x F_{v_x} + \beta_y F_{v_y})\, dx\, dy \ .\end{aligned}$$

Consider the last two terms of the first integral above:

$$A = \iint_D (\alpha_x F_{u_x} + \alpha_y F_{u_y})\, dx\, dy \ .$$

From the rule of differentiation of a product we have

$$\frac{\partial}{\partial x}(\alpha F_{u_x}) = \alpha_x F_{u_x} + \alpha \frac{\partial F_{u_x}}{\partial x} \quad \text{so that} \quad \alpha_x F_{u_x} = \frac{\partial}{\partial x}(\alpha F_{u_x}) - \alpha \frac{\partial F_{u_x}}{\partial x}$$

and similarly

$$\alpha_y F_{u_y} = \frac{\partial}{\partial y}(\alpha F_{u_y}) - \alpha \frac{\partial F_{u_y}}{\partial y} \ .$$

Then

$$A = \iint_D \left[\frac{\partial}{\partial x}(\alpha F_{u_x}) + \frac{\partial}{\partial y}(\alpha F_{u_y}) - \alpha \left(\frac{\partial F_{u_x}}{\partial x} + \frac{\partial F_{u_y}}{\partial y} \right) \right] dx\, dy \ .$$

The two-dimensional equivalent of integration by parts is Green's theorem:

$$\iint_D \left(\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y} \right) dx\, dy = \int_{\partial D} Q dy - P dx \ .$$

If we let

$$P = \alpha F_{u_y} \quad \text{and} \quad Q = \alpha F_{u_x} \ ,$$

this yields

$$A = \int_{\partial D} \alpha (F_{u_x} dy - F_{u_y} dx) \ - \ \iint_D \alpha \left(\frac{\partial F_{u_x}}{\partial x} + \frac{\partial F_{u_y}}{\partial y} \right) dx\, dy$$

and

$$\frac{\partial I}{\partial \delta} = \iint_D (\alpha F_u + A)\, dx\, dy = \int_{\partial D} \alpha (F_{u_x} dy - F_{u_y} dx) \ + \ \iint_D \alpha \left(F_u - \frac{\partial F_{u_x}}{\partial x} - \frac{\partial F_{u_y}}{\partial y} \right) dx\, dy \ .$$

Recall that we want this derivative to be zero for *all* test functions α. If we pick in particular a function α that is nonzero only on the boundary ∂D, then the last integral above vanishes (since ∂D has measure zero in D). So we need

$$F_{u_x} dy = F_{u_y} dx$$

for the first of the two integrals above to vanish. If we define a parameter s along ∂D, this also implies

$$F_{u_x} \frac{dy}{ds} = F_{u_y} \frac{dx}{ds} \ .$$

This is the first of two *natural boundary conditions*.[1] Thus, we obtain

$$\iint_D \alpha \left(F_u - \frac{\partial F_{u_x}}{\partial x} - \frac{\partial F_{u_y}}{\partial y} \right) dx\, dy = 0 \ .$$

For this equation to hold for all α, the function in parentheses must be identically zero:

$$F_u - \frac{\partial F_{u_x}}{\partial x} - \frac{\partial F_{u_y}}{\partial y} = 0 \ .$$

This is the first of the two Euler equations in the theorem. The same reasoning starting from $\frac{\partial I}{\partial \epsilon} = 0$ yields the second equation, together with the other natural boundary condition:

$$F_v - \frac{\partial F_{v_x}}{\partial x} - \frac{\partial F_{v_y}}{\partial y} = 0 \quad \text{and} \quad F_{v_x} \frac{dy}{ds} = F_{v_y} \frac{dx}{ds} \ .$$

References

[1] B. K. P. Horn. *Robot Vision*. Mc Graw-Hill, New York, New York, 1986.

[1] If we had *imposed* boundary conditions on u instead, then the test function α would have to be zero on the boundary, for otherwise $u + \delta\alpha$ would not satisfy the conditions. Either way, the integral on ∂D vanishes, although for different reasons in the two cases.