COMPSCI 102: Introduction to Discrete Mathematics *Homework 4*

Due: Monday, October 30, 2007. Most problems taken from Rosen.

1. (*5 points*) A coin is flipped until it comes up heads. The sample space of the experiment is $\{T, HT, HHT, HHHT, HHHHT, \ldots\}$. What is the expected number of flips required for the experiment to end?

2. (*5 points*) A fair coin is flipped n times producing a sequence $s_1, \ldots, s_n$ of heads and tails, such as $HTTHH$ for $n = 5$. Let X denote the number of positions at which two heads appear in a row, i.e., the number of positions i, where $1 \leq i < n$, such that $s_i = H$ and $s_{i+1} = H$. What is $E(X)$?

3. (*5 points*) As in the previous problem, let X denote the number of positions at which two heads appear in a row. Write X as the sum of two independent random variables.

4. (*5 points*) Using the method of repeated halving, convert the following integers from decimal notation to binary notation.
   (a) 321
   (b) 1023
   (c) 100632

5. (*10 points)* Devise an algorithm for converting the sum of three n-bit numbers into the sum of two $(n+1)$-bit numbers without performing any carries from one bit position to the next. *Hint:* Consider each ibput bit position independently.

6. (*5 points*) The value of the Euler ϕ-function at the positive integer n is defined to be the number of positive integers less than or equal to n that are relatively prime to n. Show that n is prime if and only if $\phi(n) = n - 1$.

7. (*5 points*) Show that a positive integer is divisible by 11 if and only if the difference of the sum of its decimal digits in even-numbered positions and the sum of its decimal digits in odd-numbered positions is divisible by 11.

8. (*5 points*) Show that if a, b, c, and d, are integers such that $a|c$ and $b|d$, then $ab|cd$.

9. (*5 points*) How many zeros are there at the end of 100!?

10. (*5 points*) We call a positive integer **perfect** if it equals the sum of its positive divisors other than itself.
    (a) Show that 6 and 28 are perfect.
    (b) Show that $2^{p-1}(2^p - 1)$ is a perfect number when $2^p - 1$ is prime.

11. (*5 points*) Determine whether each of the following are Mersenne primes.
    (a) $2^7 - 1$
    (b) $2^9 - 1$
    (c) $2^{11} - 1$
    (d) $2^{13} - 1$

12. (*5 points*) Show that if a and b are positive integers, then $ab = \gcd(a, b) \cdot \operatorname{lcm}(a, b)$.

13. (*5 points*) Use the Euclidean algorithm to find

    (a) $\gcd(1, 5)$

    (b) $\gcd(100, 101)$

    (c) $\gcd(123, 277)$

    (d) $\gcd(1529, 14039)$

    (e) $\gcd(1529, 14038)$

    (f) $\gcd(11111, 111111)$

14. (*10 points*) Prove that the following variant of the Euclidean algorithm correctly computes $\gcd(a, b)$, where a and b are positive integers. If $a = 0$ or $b = 0$ then $\gcd(a, b) = \max(a, b)$. Otherwise, $\gcd(a, b) = \gcd(\max(a, b) - \min(a, b), \min(a, b))$.

15. (*5 points*) Solve the following continued fractions.

    (a) $\frac{1}{2+\frac{1}{2+\frac{1}{2+\cdots}}}$

    (b) $\frac{1}{3+\frac{1}{3+\frac{1}{3+\cdots}}}$

16. (*15 points*) You've seen that the number $\phi = \frac{1+\sqrt{5}}{2}$ has the continued fraction expansion $1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\cdots}}}$. It turns out that continued fractions are very closely related to the Euclidean algorithm. In this problem, you'll show that when we run the Euclidean algorithm on the Fibonacci numbers (which are very closely related to the number ϕ), you get behaviour that is somewhat similar.

    a) *(5 points)* For $n \geq 1$, how many divisions does Euclid's algorithm take to compute the GCD of $Fib(n + 1)$ and $Fib(n)$? As usual, you need to prove your answer.

       It may be useful to observe that, whenever we are taking one number modulo another in the algorithm, the quotient (ignoring the remainder, which you use for the next step in the algorithm) is 1 — just like how the terms of the continued fraction are all 1. That is no coincidence; it is a consequence of how the Euclidean algorithm and continued fractions are closely connected to each other.

    b) *(10 points)* Suppose $g(n)$ is the answer for (a). Show that if $A > B$ are positive integers and Euclid's algorithm takes $g(n)$ divisions to compute $GCD(A, B)$, then $A \geq Fib(n + 1)$ and $B \geq Fib(n)$.