CPS 102: Discrete Mathematics	**Instructor:** Bruce Maggs
Assignment 6	**Due:** Friday December 4, 2009

1 Trees (10 points)

Show that in any n-node undirected tree, there is a single node (called a *separator*) whose removal partitions the nodes into two sets A and B such that each set has at most $2n/3$ nodes, and no edge connects a node in A with a node in B. Hint: start at any node in the tree, and "walk" towards a better separator, if necessary.

2 Trees (10 points)

[From Rosen] Suppose that $d_1, d_2, \ldots, d_n$ are n positive integers with sum $2(n-1)$. Show that there is a tree which has n vertices so that the degrees of these vertices are $d_1, d_2, \ldots, d_n$.

3 Minimum Spanning Trees (10 points)

Show that if every edge in an undirected graph G has a distinct weight then there is a unique minimum-weight spanning tree T for G. Hint: study the lecture on minimum-weight spanning trees.

4 Graphs (5 points)

Show that if the maximum degree of any node in an undirected graph G is δ, then the nodes of G can be colored using $\delta + 1$ colors so that no edge connects two nodes of the same color. Hint: you are thinking too hard if you need a hint on this one.

5 Prefix Sum (10 points)

In a *segmented* prefix sum, there is an input sequence $A = a_0, a_1, \ldots, a_{n-1}$ and a binary associative operator $\oplus$ over the elements in A, and a second sequence $B = b_0, b_1, \ldots, b_{n-1}$, where each element of B is either 0 or 1. The second sequence, B, is used to segment A into smaller subsequences such that an independent prefix sum is performed for each subsequence. In particular, if $b_i = 1$, then a new subsequence starts at a_i. As an example consider the sequences $A = 1, 3, 6, 2, 5, 3$ and $B = 1, 0, 0, 1, 0, 0$. If $\oplus$ is integer addition, then the output of the segmentented prefix sum is $1, 4, 10, 2, 7, 10$. Show that segmented prefix sum is just a special case of prefix sum. Do this by showing how to map the inputs and operator of a segmented prefix sum to the inputs and operator of a standard prefix sum. Be sure to show that your operator is associative.

6 Graphs (10 points)

A graph with vertices numbers 1 through n can be represented using two sequences A and B as follows. The first sequence, A, lists the neighbors of each vertex, one after another. For example, for the complete graph on three vertices, A is $2, 3, 1, 3, 1, 2$. The second sequence segments the first so that there is a distinct subsequence (list of neighbors) for each vertex. For the complete graph on three vertices, B is $1, 0, 1, 0, 1, 0$. Show how to compute, for each vertex, the largest numbered neighbor. Your algorithm may perform a constant number of segmented (or standard) prefix sums, and a constant (*not linear*) number of additional operations.

7 Long Division (10 points)

Show that every rational number has a repeated decimal representation, i.e., a representation of the form $1.45\overline{69} = 1.45696969696969...$ Hint: analyze the standard long-division algorithm and use the pigeonhole principle.

8 Group Theory (5 points)

Prove that the set of rational numbers is closed under addition and multiplication.

9 Rational Numbers (5 points)

Show that any number that can be represented using a repeated decimal representation is rational.

10 Irrational Numbers (5 points)

Prove that a real number is irrational if and only if it does not have a repeated decimial representation.

11 Cardinality (5 points)

Prove that the set of real numbers has the same cardinality as the set of complex numbers. Hint: consider interleaving digits.

12 Contradiction (5 points)

What's wrong with the following definition? S is the set of all sets that do not contain themselves.

13 Undecidability (10 points)

Prove that the set of all triples (P, I, l) such that line l of program P is executed when run on input I is undecidable.