Extensive-form games with perfect information

- Players do not move simultaneously
- When moving, each player is aware of all the previous moves (perfect information)
- A (pure) strategy for player $i$ is a mapping from player $i$’s nodes to actions

Leaves of the tree show player 1’s utility first, then player 2’s utility
Backward induction

• When we know what will happen at each of a node’s children, we can decide the best action for the player who is moving at that node.
A limitation of backward induction

- If there are ties, then how they are broken affects what happens higher up in the tree
- Multiple equilibria…
Conversion from extensive to normal form

LR = Left if 1 moves Left, Right if 1 moves Right; etc.

<table>
<thead>
<tr>
<th></th>
<th>LL</th>
<th>LR</th>
<th>RL</th>
<th>RR</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>L</strong></td>
<td>3, 2</td>
<td>3, 2</td>
<td>2, 3</td>
<td>2, 3</td>
</tr>
<tr>
<td><strong>R</strong></td>
<td>4, 1</td>
<td>0, 1</td>
<td>4, 1</td>
<td>0, 1</td>
</tr>
</tbody>
</table>

• Nash equilibria of this normal-form game include (R, LL), (R, RL), (L, RR) + infinitely many mixed-strategy equilibria

• In general, normal form can have exponentially many strategies
Converting the first game to normal form

- Pure-strategy Nash equilibria of this game are (LL, LR), (LR, LR), (RL, LL), (RR, LL)
- But the only backward induction solution is (RL, LL)
Subgame perfect equilibrium

- Each node in a (perfect-information) game tree, together with the remainder of the game after that node is reached, is called a subgame.
- A strategy profile is a subgame perfect equilibrium if it is an equilibrium for every subgame.

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<th>RL</th>
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</tr>
</thead>
<tbody>
<tr>
<td>LL</td>
<td>2, 4</td>
<td>2, 4</td>
<td>5, 3</td>
<td>5, 3</td>
</tr>
<tr>
<td>LR</td>
<td>2, 4</td>
<td>2, 4</td>
<td>5, 3</td>
<td>5, 3</td>
</tr>
<tr>
<td>RL</td>
<td>3, 2</td>
<td>1, 0</td>
<td>3, 2</td>
<td>1, 0</td>
</tr>
<tr>
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</tbody>
</table>

- (RR, LL) and (LR, LR) are not subgame perfect equilibria because (*R, **) is not an equilibrium.
- (LL, LR) is not subgame perfect because (*L, *R) is not an equilibrium.
- *R is not a credible threat.
Imperfect information

- Dotted lines indicate that a player cannot distinguish between two (or more) states
  - A set of states that are connected by dotted lines is called an information set

- Reflected in the normal-form representation

- Any normal-form game can be transformed into an imperfect-information extensive-form game this way
A poker-like game

1 gets King 1 gets Jack
bet bet stay stay
call fold call fold call fold call fold

player 1

1 gets Jack
bet stay

player 2

player 2

call fold call fold

2/3 1/3
cc cf fc ff

1/3 bb
0, 0 0, 0 1, -1 1, -1
2/3 bs .5, -.5 1.5, -1.5 0, 0 1, -1
1/3 sb -.5, .5 -5, 5 1, -1 1, -1
1/3 ss 0, 0 1, -1 0, 0 1, -1
Subgame perfection and imperfect information

- How should we extend the notion of subgame perfection to games of imperfect information?

- We cannot expect Player 2 to play Right after Player 1 plays Left, and Left after Player 1 plays Right, because of the information set.

- Let us say that a subtree is a subgame only if there are no information sets that connect the subtree to parts outside the subtree.
Subgame perfection and imperfect information...

- One of the Nash equilibria is: (R, RR)
- Also subgame perfect (the only subgames are the whole game, and the subgame after Player 1 moves Right)
- But it is not reasonable to believe that Player 2 will move Right after Player 1 moves Left/Middle (not a credible threat)
- There exist more sophisticated refinements of Nash equilibrium that rule out such behavior
Computing equilibria in the extensive form

• Can just use normal-form representation
  – Misses issues of subgame perfection, etc.
• Another problem: there are exponentially many pure strategies, so normal form is exponentially larger
  – Even given polynomial-time algorithms for normal form, time would still be exponential in the size of the extensive form
• There are other techniques that reason directly over the extensive form and scale much better
  – E.g., using the sequence form of the game
Commitment

• Consider the following (normal-form) game:

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<th></th>
<th>2, 1</th>
<th>4, 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 0</td>
<td>3, 1</td>
<td></td>
</tr>
</tbody>
</table>

• How should this game be played?
• Now suppose the game is played as follows:
  – Player 1 *commits* to playing one of the rows,
  – Player 2 observes the commitment and then chooses a column

• What is the optimal strategy for player 1?
• What if 1 can commit to a *mixed* strategy?
Commitment as an extensive-form game

- For the case of committing to a pure strategy:
Commitment as an extensive-form game

- For the case of committing to a mixed strategy:

- Infinite-size game; computationally impractical to reason with the extensive form here.
Solving for the optimal mixed strategy to commit to

[Conitzer & Sandholm 2006; see also: von Stengel & Zamir 2004, Letchford, Conitzer, Munagala 2009]

• For every column $t$ separately, we will solve separately for the best mixed row strategy (defined by $p_s$) that induces player 2 to play $t$

• maximize $\Sigma_s p_s u_1(s, t)$

• subject to
  
  for any $t'$, $\Sigma_s p_s u_2(s, t) \geq \Sigma_s p_s u_2(s, t')$

  $\Sigma_s p_s = 1$

• (May be infeasible, e.g., if $t$ is strictly dominated)

• Pick the $t$ that is best for player 1
Visualization

\[
\begin{array}{c|ccc}
 & L & C & R \\
\hline
U & 0,1 & 1,0 & 0,0 \\
M & 4,0 & 0,1 & 0,0 \\
D & 0,0 & 1,0 & 1,1 \\
\end{array}
\]