CPS 296.1
Voting and social choice

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Voting over alternatives

voting rule (mechanism) determines winner based on votes

• Can vote over other things too
  – Where to go for dinner tonight, other joint plans, …
Voting (rank aggregation)

• Set of m candidates (aka. alternatives, outcomes)
• n voters; each voter ranks all the candidates
  – E.g., for set of candidates {a, b, c, d}, one possible vote is b > a > d > c
  – Submitted ranking is called a vote

• A voting rule takes as input a vector of votes (submitted by the voters), and as output produces either:
  – the winning candidate, or
  – an aggregate ranking of all candidates

• Can vote over just about anything
  – political representatives, award nominees, where to go for dinner tonight, joint plans, allocations of tasks/resources, …
  – Also can consider other applications: e.g., aggregating search engine’s rankings into a single ranking
Example voting rules

• **Scoring rules** are defined by a vector \((a_1, a_2, \ldots, a_m)\); being ranked \(i\)th in a vote gives the candidate \(a_i\) points
  
  – **Plurality** is defined by \((1, 0, 0, \ldots, 0)\) (winner is candidate that is ranked first most often)
  
  – **Veto** (or anti-plurality) is defined by \((1, 1, \ldots, 1, 0)\) (winner is candidate that is ranked last the least often)
  
  – **Borda** is defined by \((m-1, m-2, \ldots, 0)\)

• **Plurality with (2-candidate) runoff**: top two candidates in terms of plurality score proceed to runoff; whichever is ranked higher than the other by more voters, wins

• **Single Transferable Vote (STV, aka. Instant Runoff)**: candidate with lowest plurality score drops out; if you voted for that candidate, your vote transfers to the next (live) candidate on your list; repeat until one candidate remains

• Similar runoffs can be defined for rules other than plurality
Pairwise elections

two votes prefer Obama to McCain

two votes prefer Obama to Nader

two votes prefer Nader to McCain

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Condorcet cycles

two votes prefer McCain to Obama

two votes prefer Obama to Nader

two votes prefer Nader to McCain

“weird” preferences
Voting rules based on pairwise elections

- **Copeland**: candidate gets two points for each pairwise election it wins, one point for each pairwise election it ties
- **Maximin** (aka. **Simpson**): candidate whose worst pairwise result is the best wins
- **Slater**: create an overall ranking of the candidates that is inconsistent with as few pairwise elections as possible
  - NP-hard!
- **Cup/pairwise elimination**: pair candidates, losers of pairwise elections drop out, repeat
• **Kemeny**: create an overall ranking of the candidates that has as few *disagreements* as possible (where a disagreement is with a vote on a pair of candidates)
  – NP-hard!
• **Bucklin**: start with $k=1$ and increase $k$ gradually until some candidate is among the top $k$ candidates in more than half the votes; that candidate wins
• **Approval** (not a ranking-based rule): every voter labels each candidate as approved or disapproved, candidate with the most approvals wins
• … how do we choose a rule from all of these rules?
• How do we know that there does not exist another, “perfect” rule?
• Let us look at some *criteria* that we would like our voting rule to satisfy
Condorcet criterion

- A candidate is the Condorcet winner if it wins all of its pairwise elections
- Does not always exist…
- … but the Condorcet criterion says that if it does exist, it should win

- Many rules do not satisfy this
- E.g. for plurality:
  - \( b > a > c > d \)
  - \( c > a > b > d \)
  - \( d > a > b > c \)
- \( a \) is the Condorcet winner, but it does not win under plurality
**Majority criterion**

- If a candidate is ranked first by most votes, that candidate should win
  - Relationship to Condorcet criterion?

- Some rules do not even satisfy this
- E.g. Borda:
  - $a > b > c > d > e$
  - $a > b > c > d > e$
  - $c > b > d > e > a$

- $a$ is the majority winner, but it does not win under Borda
Monotonicity criteria

- Informally, monotonicity means that “ranking a candidate higher should help that candidate,” but there are multiple nonequivalent definitions.
- A **weak** monotonicity requirement: if
  - candidate w wins for the current votes,
  - we then improve the position of w in some of the votes and leave everything else the same,
  then w should still win.
- E.g., STV does not satisfy this:
  - 7 votes b > c > a
  - 7 votes a > b > c
  - 6 votes c > a > b
- c drops out first, its votes transfer to a, a wins
- But if 2 votes b > c > a change to a > b > c, b drops out first, its 5 votes transfer to c, and c wins
Monotonicity criteria…

• A strong monotonicity requirement: if
  – candidate w wins for the current votes,
  – we then change the votes in such a way that for each vote, if a candidate c was ranked below w originally, c is still ranked below w in the new vote
  then w should still win.

• Note the other candidates can jump around in the vote, as long as they don’t jump ahead of w

• None of our rules satisfy this
Independence of irrelevant alternatives

• Independence of irrelevant alternatives criterion: if
  – the rule ranks a above b for the current votes,
  – we then change the votes but do not change which is ahead between a and b in each vote
then a should still be ranked ahead of b.
• None of our rules satisfy this
Arrow’s impossibility theorem [1951]

• Suppose there are at least 3 candidates
• Then there exists no rule that is simultaneously:
  – Pareto efficient (if all votes rank a above b, then the rule ranks a above b),
  – nondictatorial (there does not exist a voter such that the rule simply always copies that voter’s ranking), and
  – independent of irrelevant alternatives
Muller-Satterthwaite impossibility theorem [1977]

• Suppose there are at least 3 candidates
• Then there exists no rule that simultaneously:
  – satisfies **unanimity** (if all votes rank a first, then a should win),
  – is **nondictatorial** (there does not exist a voter such that the rule simply always selects that voter’s first candidate as the winner), and
  – is **monotone** (in the strong sense).
Manipulability

- Sometimes, a voter is better off revealing her preferences insincerely, aka. manipulating
- E.g. plurality
  - Suppose a voter prefers \( a > b > c \)
  - Also suppose she knows that the other votes are
    - 2 times \( b > c > a \)
    - 2 times \( c > a > b \)
  - Voting truthfully will lead to a tie between \( b \) and \( c \)
  - She would be better off voting e.g. \( b > a > c \), guaranteeing \( b \) wins
- All our rules are (sometimes) manipulable
Gibbard-Satterthwaite impossibility theorem

• Suppose there are at least 3 candidates
• There exists no rule that is simultaneously:
  – onto (for every candidate, there are some votes that would make that candidate win),
  – nondictatorial, and
  – nonmanipulable
Single-peaked preferences

- Suppose candidates are ordered on a line
- Every voter prefers candidates that are closer to her most preferred candidate
- Let every voter report only her most preferred candidate ("peak")
- Choose the median voter’s peak as the winner
  - This will also be the Condorcet winner
- Nonmanipulable!

\[
\begin{array}{cccccc}
\text{v}_5 & \text{v}_4 & \text{v}_2 & \text{v}_1 & \text{v}_3 \\
a_1 & a_2 & a_3 & a_4 & a_5 \\
\end{array}
\]

*Impossibility results do not necessarily hold when the space of preferences is restricted*
Pairwise election graphs

- **Pairwise election** between \(a\) and \(b\): compare how often \(a\) is ranked above \(b\) vs. how often \(b\) is ranked above \(a\)
- Graph representation: edge from winner to loser (no edge if tie), weight = margin of victory
- E.g., for votes \(a > b > c > d\), \(c > a > d > b\) this gives

```
\begin{tikzpicture}
  \node[circle,draw] (a) at (0,0) {a};
  \node[circle,draw] (b) at (1,0) {b};
  \node[circle,draw] (c) at (0,-1) {c};
  \node[circle,draw] (d) at (1,-1) {d};
  \draw[->] (a) to[bend right] node [right] {2} (b);
  \draw[->] (b) to[bend right] node [right] {2} (a);
  \draw[->] (c) to[bend right] node [right] {2} (b);
  \draw[->] (b) to[bend right] node [right] {2} (c);
\end{tikzpicture}
```
Kemeny on pairwise election graphs

• Final ranking = acyclic tournament graph
  – Edge (a, b) means a ranked above b
  – Acyclic = no cycles, tournament = edge between every pair

• Kemeny ranking seeks to minimize the total weight of the inverted edges

(pairwise election graph)

\[(b > d > c > a)\]
Slater on pairwise election graphs

- Final ranking = acyclic tournament graph
- Slater ranking seeks to minimize the number of inverted edges

**pairwise election graph**

![Pairwise election graph]

**Slater ranking**

![Slater ranking]

\[(a > b > d > c)\]
An integer program for computing Kemeny/Slater rankings

$y_{(a, b)}$ is 1 if $a$ is ranked below $b$, 0 otherwise

$w_{(a, b)}$ is the weight on edge $(a, b)$ (if it exists)

in the case of Slater, weights are always 1

minimize: $\sum_{e \in E} w_e y_e$

subject to:

for all $a, b \in V$, $y_{(a, b)} + y_{(b, a)} = 1$

for all $a, b, c \in V$, $y_{(a, b)} + y_{(b, c)} + y_{(c, a)} \geq 1$
Some computational issues in social choice

- Sometimes computing the winner/aggregate ranking is hard
  - E.g. for Kemeny and Slater rules this is NP-hard
- For some rules (e.g., STV), computing a successful manipulation is NP-hard
  - Manipulation being hard is a good thing (circumventing Gibbard-Satterthwaite?)… But would like something stronger than NP-hardness
  - Also: work on the complexity of controlling the outcome of an election by influencing the list of candidates/schedule of the Cup rule/etc.

- Preference elicitation:
  - We may not want to force each voter to rank all candidates;
  - Rather, we can selectively query voters for parts of their ranking, according to some algorithm, to obtain a good aggregate outcome

- Combinatorial alternative spaces:
  - Suppose there are multiple interrelated issues that each need a decision
  - Exponentially sized alternative spaces

- Different models such as ranking webpages (pages “vote” on each other by linking)