Worst-Case Optimal Redistribution of VCG Payments in Multi-Unit Auctions

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This talk covers material from:

Guo and Conitzer, “Worst-Case Optimal Redistribution of VCG Payments in Multi-Unit Auctions” GEB 2009
Second-price (Vickrey) auction

- 1st bidder pays 3

1st bidder's valuation: $v(\text{item}) = 4$

2nd bidder's valuation: $v(\text{item}) = 3$

3rd bidder's valuation: $v(\text{item}) = 2$

- 1st bidder receives item

1st bidder pays 3

2nd bidder's valuation: $v(\text{item}) = 4$

3rd bidder's valuation: $v(\text{item}) = 2$
Vickrey auction without a seller

$v(\emptyset) = 2$
$v(\emptyset) = 4$
$v(\emptyset) = 3$

pays 3
(money wasted!)
Can we redistribute the payment?

Idea: give everyone $1/n$ of the payment

$$v(\text{Alice}) = 2$$
$$v(\text{Bob}) = 4$$
$$v(\text{Charlie}) = 3$$

- Alice receives 1
- Bob pays 3
- Charlie receives 1

**not** incentive compatible

Bidding higher can increase your redistribution payment
Incentive compatible redistribution

[Bailey 97, Porter et al. 04, Cavallo 06]

Idea: give everyone $1/n$ of second-highest other bid

$v(\text{first}) = 2$
$v(\text{second}) = 4$
$v(\text{third}) = 3$

receives 1
pays 3
receives 2/3
receives 2/3

$2/3$ wasted (22%)

incentive compatible

Your redistribution does not depend on your bid; incentives are the same as in Vickrey
Bailey-Cavallo mechanism...

- Bids: $V_1 \geq V_2 \geq V_3 \geq \ldots \geq V_n \geq 0$
- First run Vickrey auction
- Payment is $V_2$
- First two bidders receive $V_3/n$
- Remaining bidders receive $V_2/n$
- Total redistributed: $2V_3/n+(n-2)V_2/n$

Can we do better?
Desirable properties

- Incentive compatibility
- Individual rationality: bidder’s utility always nonnegative
- Efficiency: bidder with highest valuation gets item
- Non-deficit: sum of payments is nonnegative
  - i.e. total VCG payment ≥ total redistribution
- (Strong) budget balance: sum of payments is zero
  - i.e. total VCG payment = total redistribution

Impossible to get all

- We sacrifice budget balance
  - Try to get approximate budget balance
- Other work sacrifices: incentive compatibility [Parkes 01], efficiency [Faltings 04], non-deficit [Bailey 97], budget balance [Cavallo 06]
Another redistribution mechanism

- Bids: $V_1 \geq V_2 \geq V_3 \geq V_4 \geq \ldots \geq V_n \geq 0$
- First run Vickrey
- Redistribution:
  Receive $1/(n-2)$ * second-highest other bid, $- 2/[(n-2)(n-3)]$ third-highest other bid
- Total redistributed:
  $V_2 - 6V_4/[(n-2)(n-3)]$
- Efficient & incentive compatible
- Individually rational & non-deficit (for large enough $n$)

\[
\begin{align*}
R_1 &= \frac{V_3}{(n-2)} - 2/[(n-2)(n-3)]V_4 \\
R_2 &= \frac{V_3}{(n-2)} - 2/[(n-2)(n-3)]V_4 \\
R_3 &= \frac{V_2}{(n-2)} - 2/[(n-2)(n-3)]V_4 \\
R_4 &= \frac{V_2}{(n-2)} - 2/[(n-2)(n-3)]V_3 \\
&\vdots \\
R_{n-1} &= \frac{V_2}{(n-2)} - 2/[(n-2)(n-3)]V_3 \\
R_n &= \frac{V_2}{(n-2)} - 2/[(n-2)(n-3)]V_3
\end{align*}
\]
Comparing redistributions

- Bailey-Cavallo: \( \Sigma R_i = 2V_3/n + (n-2)V_2/n \)
- Second mechanism: \( \Sigma R_i = V_2 - 6V_4/[(n-2)(n-3)] \)
- Sometimes the first mechanism redistributes more
- Sometimes the second redistributes more
- Both redistribute 100% in some cases
- What about the worst case?
- Bailey-Cavallo worst case: \( V_3 = 0 \)
  - percentage redistributed: 1-2/n
- Second mechanism worst case: \( V_2 = V_4 \)
  - percentage redistributed: 1-6/[(n-2)(n-3)]
- For large enough \( n \), \( 1-6/[(n-2)(n-3)] \geq 1-2/n \), so second is better (in the worst case)
Generalization: linear redistribution mechanisms

• Run Vickrey
• Amount redistributed to bidder:
  \[ C_0 + C_1 S_1 + C_2 S_2 + \ldots + C_{n-1} S_{n-1} \]

where \( S_j \) is the \( j \)-th highest other bid

• Bailey-Cavallo: \( C_2 = 1/n \)
• Second mechanism: \( C_2 = 1/(n-2) \), \( C_3 = -2/[(n-2)(n-3)] \)

• Bidder’s redistribution does not depend on own bid, so incentive compatible
• Efficient
• Other properties?
Redistribution to each bidder

Recall: \( R = C_0 + C_1 S_1 + C_2 S_2 + \ldots + C_{n-1} S_{n-1} \)

\[
R_1 = C_0 + C_1 V_2 + C_2 V_3 + C_3 V_4 + \ldots + C_i V_{i+1} + \ldots + C_{n-1} V_n \\
R_2 = C_0 + C_1 V_1 + C_2 V_3 + C_3 V_4 + \ldots + C_i V_{i+1} + \ldots + C_{n-1} V_n \\
R_3 = C_0 + C_1 V_1 + C_2 V_2 + C_3 V_4 + \ldots + C_i V_{i+1} + \ldots + C_{n-1} V_n \\
R_4 = C_0 + C_1 V_1 + C_2 V_2 + C_3 V_3 + \ldots + C_i V_{i+1} + \ldots + C_{n-1} V_n \\
\ldots \\
R_{n-1} = C_0 + C_1 V_1 + C_2 V_2 + C_3 V_3 + \ldots + C_i V_i + \ldots + C_{n-1} V_n \\
R_n = C_0 + C_1 V_1 + C_2 V_2 + C_3 V_3 + \ldots + C_i V_i + \ldots + C_{n-1} V_{n-1} \]
Individual rationality & non-deficit

• Individual rationality:
  equivalent to

  \[ R_n = C_0 + C_1 V_1 + C_2 V_2 + C_3 V_3 + \ldots + C_i V_i + \ldots + C_{n-1} V_{n-1} \geq 0 \]
  for all \( V_1 \geq V_2 \geq V_3 \geq \ldots \geq V_{n-1} \geq 0 \)

• Non-deficit:
  \[ \sum R_i \leq V_2 \text{ for all } V_1 \geq V_2 \geq V_3 \geq \ldots \geq V_{n-1} \geq V_n \geq 0 \]
Worst-case optimal (linear) redistribution

Try to maximize worst-case redistribution %

Variables: $C_i, K$

Maximize $K$

subject to:

$R_n \geq 0$ for all $V_1 \geq V_2 \geq V_3 \geq ... \geq V_{n-1} \geq 0$

$\sum R_i \leq V_2$ for all $V_1 \geq V_2 \geq V_3 \geq ... \geq V_{n} \geq 0$

$\sum R_i \geq K \cdot V_2$ for all $V_1 \geq V_2 \geq V_3 \geq ... \geq V_{n} \geq 0$

$R_i$ as defined in previous slides
Transformation into linear program

- **Claim**: $C_0 = 0$

- **Lemma**: $Q_1 X_1 + Q_2 X_2 + Q_3 X_3 + ... + Q_k X_k \geq 0$ for all $X_1 \geq X_2 \geq ... \geq X_k \geq 0$

  is equivalent to

  $Q_1 + Q_2 + ... + Q_i \geq 0$ for $i = 1$ to $k$

- Using this lemma, can write all constraints as linear inequalities over the $C_i$
Worst-case optimal remaining %

- n=5: 27% (40%)
- n=6: 16% (33%)
- n=7: 9.5% (29%)
- n=8: 5.5% (25%)
- n=9: 3.1% (22%)
- n=10: 1.8% (20%)
- n=15: 0.085% (13%)
- n=20: 3.6 e-5 (10%)
- n=30: 5.4 e-8 (7%)

- the data in the parenthesis are for Bailey-Cavallo mechanism
m-unit auction with unit demand:
VCG (m+1th price) mechanism

\[ v(\text{frame 1}) = 2 \]
\[ v(\text{frame 2}) = 4 \]
\[ v(\text{frame 3}) = 3 \]

Incentive compatible
Our techniques can be generalized to this setting
Variables: \( C_i, K \)

Maximize \( K \)

subject to:

\[ R_n \geq 0 \text{ for all } V_1 \geq V_2 \geq V_3 \geq \ldots \geq V_{n-1} \geq 0 \]

\[ \sum R_i \leq V_2 \text{ for all } V_1 \geq V_2 \geq V_3 \geq \ldots \geq V_{n-1} \geq 0 \]

\[ \sum R_i \geq K \cdot V_2 \text{ for all } V_1 \geq V_2 \geq V_3 \geq \ldots \geq V_{n-1} \geq 0 \]

\( R_i \) as defined in previous slides

Only need to change \( V_2 \) into \( mV_{m+1} \)
Results for $m+1$th price auction

BC = Bailey-Cavallo

WO = Worst-case Optimal
Analytical characterization of WO mechanism

\[ k^* = 1 - \frac{(n-1)}{\sum_{j=m}^{n-1} \binom{n-1}{j}} \]

\[ c_i^* = \frac{(-1)^{i+m-1}(n-m)\binom{n-1}{m-1}}{i\sum_{j=m}^{n-1} \binom{n-1}{j}} \frac{1}{\binom{n-1}{i}} \sum_{j=i}^{n-1} \binom{n-1}{j} \]

for \( i = m + 1, \ldots, n - 1 \)

- Unique optimum
- Can show: for fixed \( m \), as \( n \) goes to infinity, worst-case redistribution percentage approaches 100% linearly
- Rate of convergence \( 1/2 \)
Worst-case optimality outside the linear family

- **Theorem**: The worst-case optimal **linear** redistribution mechanism is also worst-case optimal among all VCG redistribution mechanisms that are
  - deterministic,
  - anonymous,
  - incentive compatible,
  - efficient,
  - non-deficit

- Individual rationality is not mentioned
  - Sacrificing individual rationality does not help
- Not **uniquely** worst-case optimal
Remarks

• Moulin's paper “Almost budget-balanced VCG mechanisms to assign multiple objects” pursues different worst-case objective (minimize waste/efficiency)
  – Results in same mechanism in the unit-demand setting (!)
  – Different mechanism results after removing individual rationality
  – Also mentions the idea of removing non-deficit property, without solving for the actual mechanism
More general settings:

multi-unit auction with nonincreasing marginal values

- A bid consists of $m$ elements: $b_1, b_2, \ldots, b_m$
  
  
  $b_i = \text{utility}(i \text{ units}) - \text{utility}(i-1 \text{ units})$
  
  $b_1 \geq b_2 \geq \ldots \geq b_m \geq 0$
Approach

• We construct a mechanism that has the same worst-case performance as the earlier WCO mechanism.

• Multi-unit auction with unit demand is a special case of multi-unit auction with nonincreasing marginal value.

• The new mechanism is optimal in the worst case.

Construction details omitted
Even more general setting?

• If marginal values are not required to be nonincreasing, the worst-case redistribution percentage is 0

Proof by example

The original VCG mechanism is already worst-case optimal

• Same for general combinatorial auction
Undominated redistribution mechanisms
[AAMAS 08]

• Sometimes redistribution mechanisms are dominated
  – another redistribution mechanism always redistributes at least as much to each agent and sometimes more
  – WCO mechanism is dominated
• We characterized mechanisms that are undominated
• We proposed two techniques for transforming any dominated redistribution mechanisms into one that dominates it
• Experimentally, the techniques significantly improve known redistribution mechanisms

• Related paper (with Apt and Markakis) [WINE 08]: variant where other mechanism redistributes at least as much and sometimes more in total
Optimal-in-expectation redistribution mechanism [AAMAS 08]

• Goal: find optimal-in-expectation (strategy-proof) redistribution mechanism
  – Analytical solution for optimal linear mechanism (OEL)
  – Discretization methodology for getting (guaranteed) almost-optimal mechanisms

• For small cases can solve for very finely discretized mechanism

• For large cases OEL is almost optimal
Better redistribution with inefficient allocation in multi-unit auctions with unit demand [EC 08]

- Inefficient mechanisms can lead to higher welfare
  - The agents’ total efficiency is smaller when the allocation is inefficient
  - But, the total payment can also be smaller (or more can be redistributed)
  - The net effect could be an increase in the total utility
    (total utility = total efficiency - total payment)

- Goal: design competitive mechanism, against the omnipotent allocation

- By allocating inefficiently (e.g., burning units, excluding agents, partitioning), we obtain more competitive mechanisms
Thank you for your attention!