CPS 102: Discrete Mathematics	**Instructor:** Bruce Maggs
Assignment 3	**Due:** Monday, October 24^{th}, 2011

Note: You cannot discuss/collaborate with any other person other than the instructor or TA. And, when solving for any question below please show your work!

1 Counting (5 points)

A college has 10 basketball players. A team of 5 players, of which 1 is a captain, will be selected out of these 10 players. How many ways are there to form the team?

2 Probability (5 points)

What is the probability that a five-card poker hand contains the ace of hearts?

3 Counting (5 points)

All PhD students in the CS department at Duke have to receive a *Quals Pass* in four of the seven designated courses spread across three streams – *AI / Num. Analysis*: $\{CPS\ 250, CPS\ 270, CPS\ 271\}$, *Theory*: $\{CPS\ 230, CPS\ 240\}$ and *Systems*: $\{CPS\ 210, CPS\ 220\}$. However, they cannot skip a stream completely and they can only take one of $\{CPS\ 270, CPS\ 271\}$. In how many ways can the PhD students choose four courses of the designated seven?

4 Probability (5 points)

What is the conditional probability that exactly four heads appear when a fair coin is flipped five times, given that the first flip came up heads?

5 Counting (5 points)

How many bit strings of length 8 contain either four consecutive 0s or four consecutive 1s?

6 Probability (5 points)

Provide an example that shows that the variance of the sum of two random variables is not necessarily equal to the sum of their variances when the random variables are not independent.

7 Counting (5 points)

How many diagonals does a regular polygon with n sides have, where n is a positive integer with $n \geq 3$?

8 Probability (5 points)

Suppose that m and n are positive integers. What is the probability that a randomly chosen positive integer less than $m \cdot n$ is not divisible by either m or n? *Note: You are allowed to use well known standard functions like GCD(m,n).*

9 Counting (5 points)

How many ways are there to seat six people around a circular table, where seatings are considered to be identical if they can be obtained from each other by rotating the table?

10 Probability (5 points)

Prof. Maggs plans to have only multiple choice questions for the next quiz to make it easy on the students. However, the TA does not want a student to pick answers at random and get high scores. In fact, the TA wants the *expected score* of such students to be zero. What do you think he should do?

11 Counting (5 points)

There are 6 boxes numbered $1, 2, ...6$. Each box is to be filled up either with a red or a green ball in such a way that at least 1 box contains a green ball and the boxes containing green balls are consecutively numbered. Find the total number of ways the boxes can be packed.

12 Probability (5 points)

Show that if E and F are independent events, then $\bar{E}$ and $\bar{F}$ are independent events.

13 Counting (5 points)

Prove that at a party where there are at least two people, there are two people who know the same number of other people there.

14 Probability (5 points)

Shuffle a standard 52-card deck, and deal out a hand of 13 cards. You get 4 points for each Ace in the hand, 3 points for each King, 2 for each Queen, 1 for each Jack, and nothing for the other cards. Let the random variable X denote the total number of points in your hand. Calculate the probability of ending up with a hand worth zero points.

15 Counting (5 points)

Determine how many ways there are to assign a birth date to each of m people in a classroom, so that we end up with exactly 3 different birthdays among them all.

16 Probability (5 points)

Prof. Maggs is a generous person when it comes to grading. This time instead of a final exam for *CPS 102*, he has devised an amazing technique to grade students. He would choose a S from the set $M : \{1, 2, ..., 100\}$ such that numbers in the range $[1, 60]$ are chosen uniformly at random with a probability of $1/3$ and $[61, 100]$ with a probability of $2/3$. He would then put S into a folder and $2 \cdot S$ into another identical one. On the day of the exam all you have to do is choose a folder at random. Prof. Maggs would then show your score in the final (no exams!) by revealing the number S in the folder[1]. You have the option of either accepting this score or trying your luck with the other *unopened* folder. Suppose you get 70 from the initial folder of your choice, would you try the other folder? What is your expected score if you don't stick to your initial choice?

17 Counting (5 points)

Show that if there are 101 people of different heights standing in a line, it is possible to find 11 (not necessarily consecutive) people in the order they are standing in the line with heights that are either increasing or decreasing. *Hint: You can use Pigeon Hole principle and construct a proof by contradiction.*

18 Probability (5 points)

What is the expected value when a \$1 lottery ticket is bought in which the purchaser wins exactly \$10 million if the ticket contains the six winning numbers chosen from the set $\{1, 2, 3, ..., 50\}$ and the purchaser wins nothing otherwise?

19 Counting (5 points)

How many ways are there for a horse race with four horses to finish if ties are possible? (Note that since ties are allowed, any number of the four horses may tie.)

[1] The acute reader shall note that Prof. Maggs is indeed generous as a score of $0 \notin M$

20 Probability (5 points)

Suppose that instead of three doors, there are four doors in the *Monty Hall* puzzle. What is the probability that you win by not changing once the host, who knows what is behind each door, opens a losing door and gives you the chance to change doors? What is the probability that you win by changing the door you select to one of the two remaining doors among the three that you did not select?