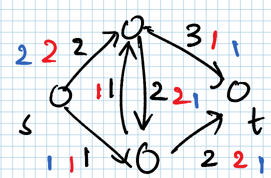
Network Flows



Graph $G = (V, A)$ [for undirected graphs $G = (V, E)$]
Edge capacity : u_e for edge e [edges also called (x,y)]
Source s, sink t

Raw flow : r_{xy} (≥ 0)
- Flow conservation: $\sum_y f_{xy} = \sum_y f_{yx}$ (for all $x \neq s,t$)
- Capacity constraint: $f_{xy} \leq u_{xy}$

E.g.: see figure

Net flow (or simply flow) : $r_{xy} - r_{yx} = f_{xy}$ (if both edges have raw flow and $r_{xy} \geq r_{yx}$)

f_{xy}
- Flow conservation : $\sum_y f_{xy} = \sum_y f_{yx}$ ($\forall x \neq s,t$)
- Capacity constraint : $f_{xy} \leq u_{xy}$
- at most one of f_{xy} or $f_{yx} > 0$.

E.g.: see figure

Total flow (or flow value) $= \sum_y f_{sy} - \sum_x f_{xs} = f$

Lemma : $\sum_y f_{sy} - \sum_x f_{xs} = \sum_x f_{xt} - \sum_y f_{ty}$ $(= f)$

Proof :

$$\sum_{(x,y)\in E} f_{xy} - \sum_{(x,y)\in E} f_{xy} = 0$$

$$\Rightarrow \sum_y f_{sy} + \sum_{x \neq (s,t)} f_{xy} + \sum_y f_{ty} - \sum_x f_{xs} - \sum_{y \neq s,t} f_{xy} - \sum_x f_{xt} = 0$$

$$\Rightarrow \left(\sum_y f_{sy} - \sum_x f_{xs}\right) + \left(\sum_y f_{ty} - \sum_x f_{xt}\right) + \underbrace{\sum_{x \neq s,t}\left(f_{xy} - f_{yx}\right)}_{0 \text{ by flow conservation}} = 0$$

$$\Rightarrow \sum_y f_{sy} - \sum_x f_{xs} = \sum_x f_{xt} - \sum_y f_{ty}$$

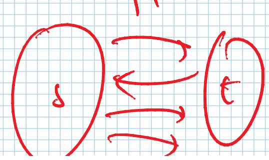
The MAXFLOW problem : find a feasible flow of maximum value

Flow decomposition and cuts

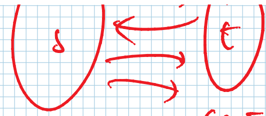
Lemma: Any feasible flow can be decomposed into $\leq m$ cycles and s-t paths with non-zero flows. Value $= \sum$ flow on paths.

Proof : find an s-t path/a cycle with non-zero flow and set its flow to maximum flow to the minimum flow on an edge of the path/cycle. Reduce the flow on every edge of the path/cycle by this amount and repeat. After removing paths, flow value $= 0$.

A cut is a bipartition of vertices/edges that go across such a bipartition. An s-t cut has s and t on different sides of the cut.



Eg: an s-t cut



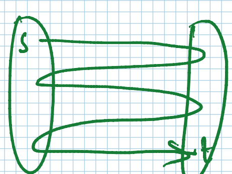
The flow across a cut $(S,\bar{S}) = \sum_{\substack{x\in S\\ y\in\bar{S}}} f_{xy} - \sum_{\substack{x\in S\\ y\in\bar{S}}} f_{yx} = f(S)$

The capacity of a cut $(S,\bar{S}) = \sum_{\substack{x\in S\\ y\in\bar{S}}} u_{xy} = u(S)$

Clearly, $f(S) \le u(S)$

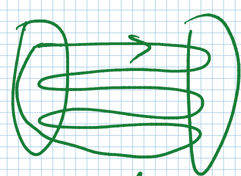
Lemma: For any s-t cut S, $f(S) = f$.

Proof: Use the flow decomposition



s-t path

$f(S) = f$ on path



cycle

$f(S) = 0$

Summing over all s-t paths, $f(S) = f$.

Corollary: $f \le \min_{S} (u(S)) =$ min s-t cut