- Algorithm is always correct but running time bounds hold in expectation)
- e.g. Randomized Quicksort
  - review quicksort
  - worst case: $T(n) = T(n-1) + O(1) \Rightarrow T(n) = \Theta(n^2)$

In randomized quicksort, pick pivot uniformly at random in each subproblem.

Analysis 1 (Backward Analysis)

In step k, # of pivots increases from k-1 to k by selecting a random pivot from the elements not selected yet. Looking backwards, in step k (steps are still indexed from the beginning), the # of pivots decreases from k to k-1.

Claim: Given a set of k pivots at the end of step k, the pivot that was selected in the kth step is uniform distributed among these k pivots.

Proof: for any two elements, their relative order of being selected as pivots is uniform.

Lemma: The expected cost in step k is $\leq 2n/k$.

Proof: The sum of costs over all the k pivots being the last one is $\leq 2n$.

Cor: The expected cost of randomized quicksort is $O(n \log n)$.

Analysis 2

Consider the comparisons that an element x_i is part of. If such a comparison splits the subproblem in $(3/4, 1/4)$ or a more balanced ratio, call it a "good" comparison; prob. of good comparison $\geq \frac{1}{2}$.

Fact: x_i is in $\leq \log_{4/3} n$ good comparisons.

Lemma: $\Pr[\#$ of comparisons for x_i till you get k good comparisons $> (1+\varepsilon)k] \leq e^{-\frac{\varepsilon^2 \cdot 2k}{3}}$.

Proof: Since comparisons being good or bad are independent, we use Chernoff bounds.

Choose $\varepsilon = \sqrt{3 \ln(4/3)}$ and $k = \log_{4/3} n$

$$\Pr[\# \text{ of comparisons for } x_i > (1+\sqrt{3\ln(4/3)})\log_{4/3} n] \leq \frac{1}{n^3}$$

$$\Pr[\exists x_i \text{ s.t. } \# \text{ of comparisons for } x_i = \omega(\log n)] \leq \frac{1}{n^2}$$

$$\Pr[\text{Running time} = O(n \log n)] \geq 1 - \frac{1}{n^2}.$$